Multi-scale Modularity in Complex Networks

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Imperial College London
100 years of living science

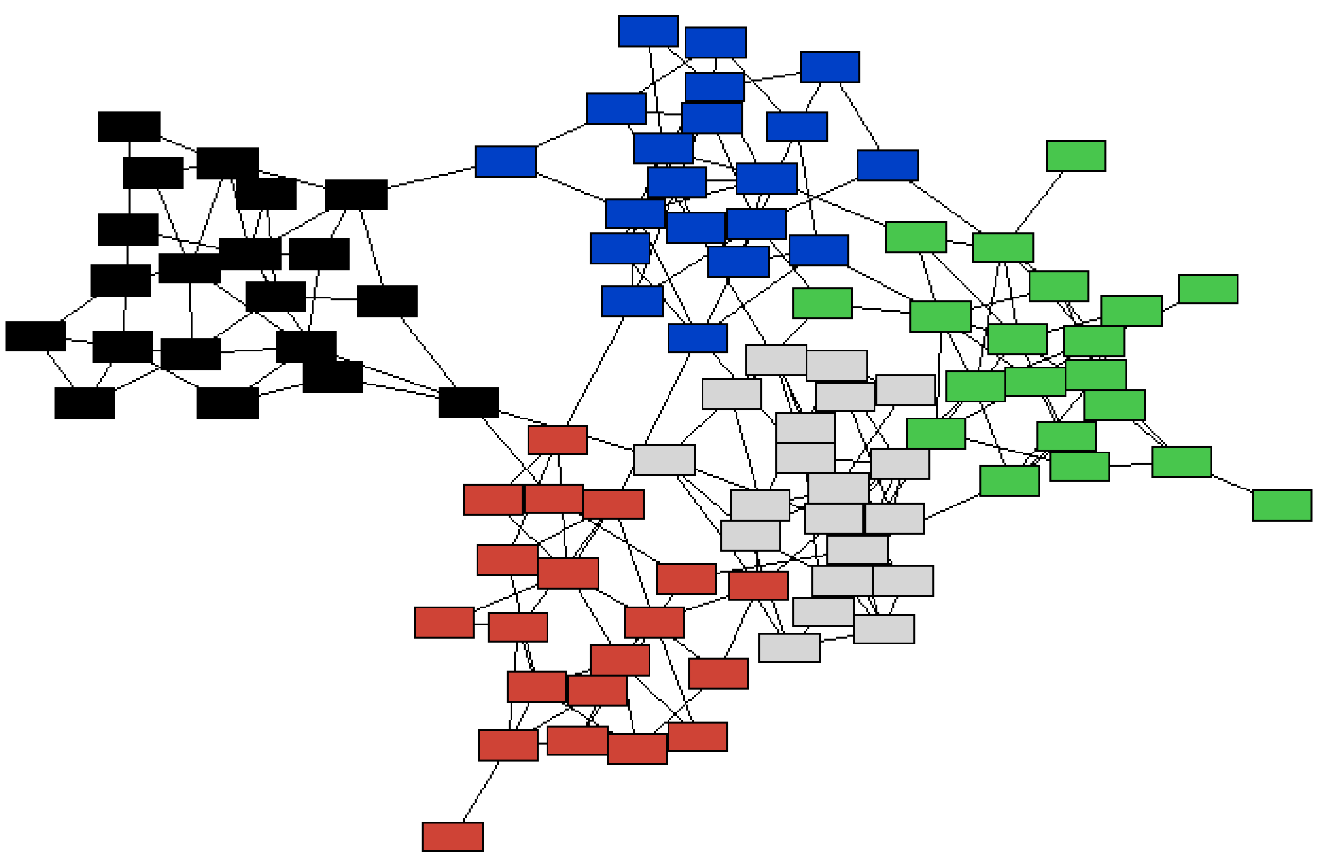
R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*
R. Lambiotte, *arXiv:1004.4268*

1. Modules and Hierarchies
2. Stability of a partition
    a. Stability vs Modularity
    b. Time as a resolution parameter
3. Optimisation and selection of the most relevant time scales/robustness
4. Dynamical networks

Modular Networks

Most networks are very inhomogeneous and are made of modules:
many links within modules and a few links between different modules

Modules=communities

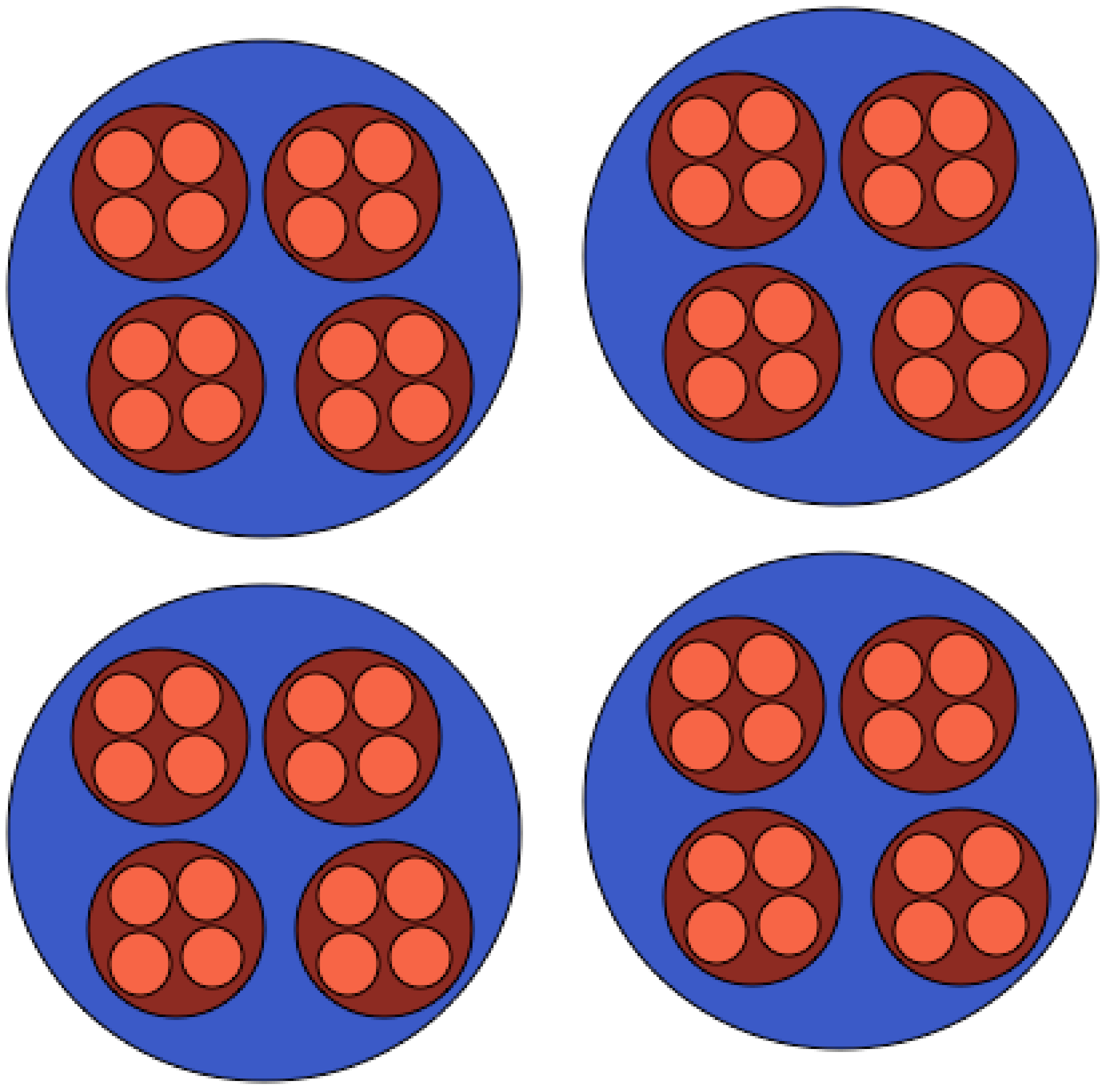


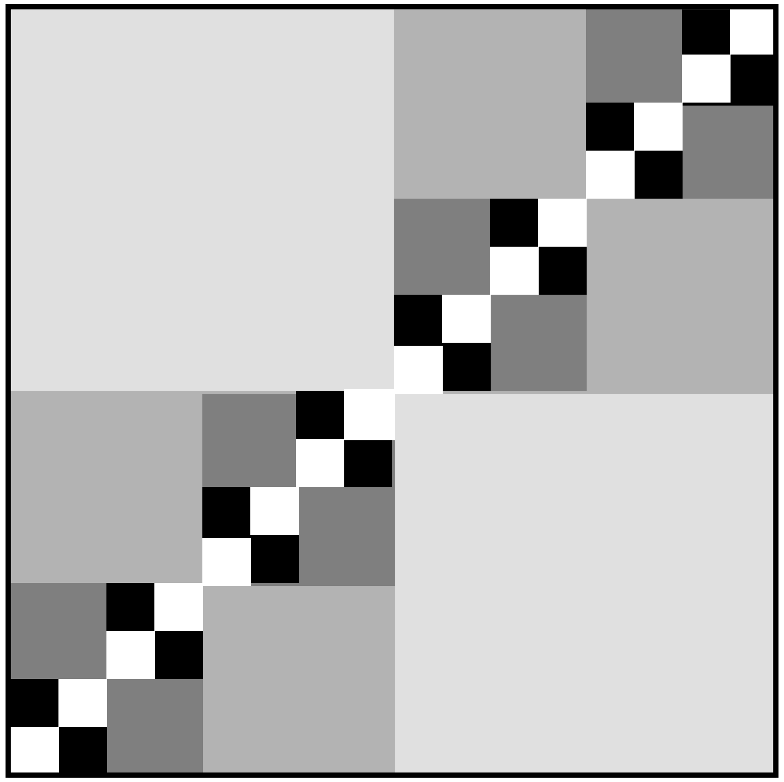
- Internet
- Power grids
- Food webs
- Metabolic networks
- Social networks
- The brain
- Etc.

Simon, H. (1962). The architecture of complexity. Proceedings of the American Philosophical Society, 106, 467–482.

Multi-scale Modular Networks

Networks have a hierarchical structure: modules within modules





Simon, H. (1962). The architecture of complexity. Proceedings of the American Philosophical Society, 106, 467–482.

Modular Networks and dynamics

Is it possible to uncover those modules/hierarchies in large networks?

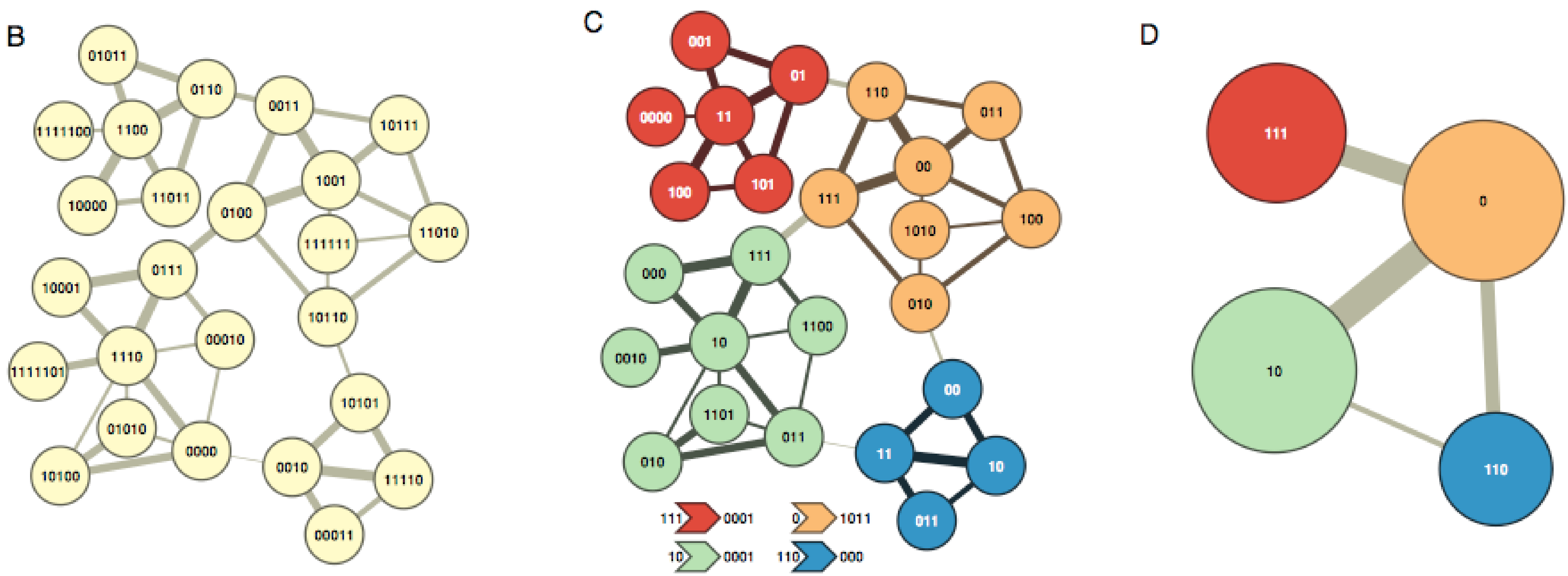
Hundreds of heuristics to optimise modularity.

How does such modularity affect dynamics?

A. Arenas, A. Diaz-Guilera and C.J. Pérez-Vicente (*Phys. Rev. Lett.*, 2006).

Modular Networks

Uncovering communities/modules helps to understand the structure of the network, to uncover similar nodes and to draw a readable map of the network (when N is large).



Find a partition of the network into communities

Coarse-grained description

Martin Rosvall and Carl T. Bergstrom, PNAS 105, 1118 –1123 (2008)

Modular Networks and dynamics

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Modular Networks and dynamics

Is it possible to uncover those modules/hierarchies in large networks?

Hundreds of heuristics to optimise modularity.

Is it possible to use dynamics to characterize (and uncover?) the modular structure of a network?

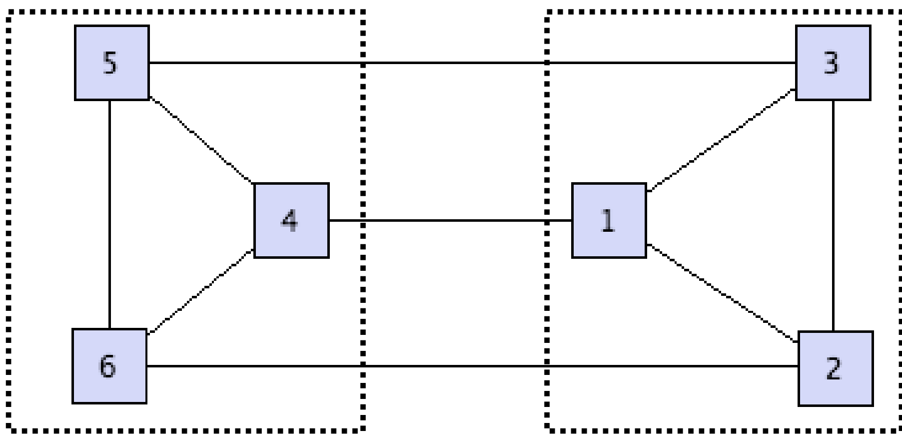
e.g. Walktrap (RW exploration), Rosvall and Bergstrom (PNAS, 2008), Fiedler (Laplacian spectrum)

How does such modularity affect dynamics?

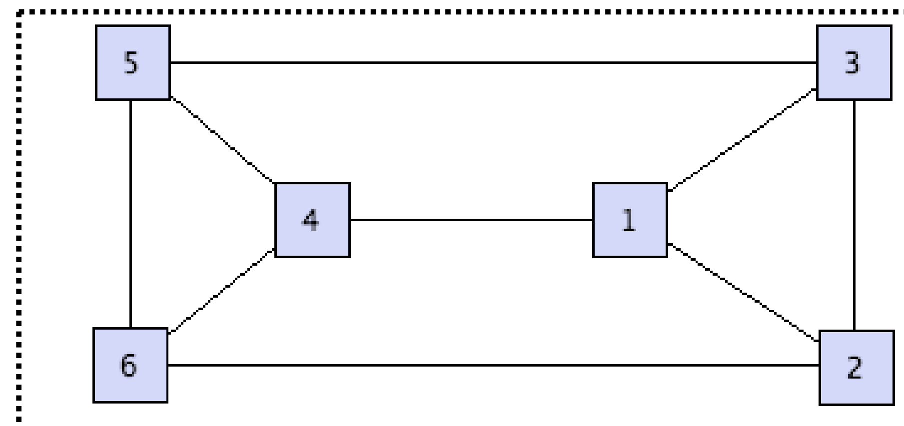
A. Arenas, A. Diaz-Guilera and C.J. Pérez-Vicente (*Phys. Rev. Lett.*, 2006).

Quality of a partition

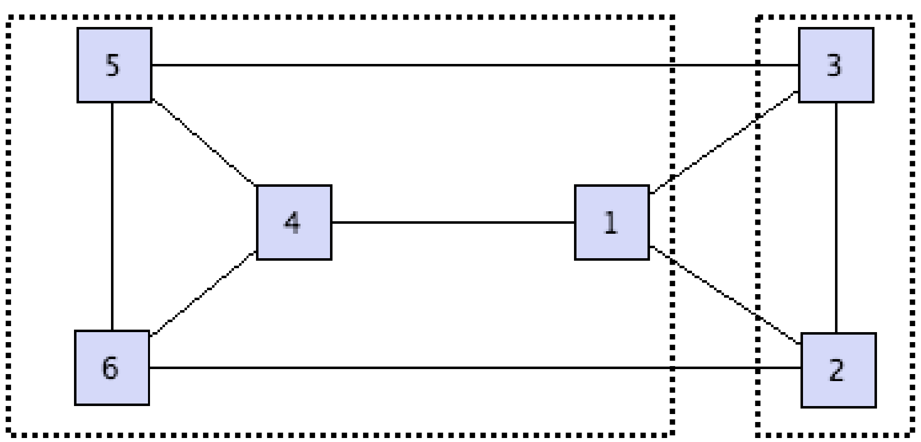
What is the best partition of a network into modules?



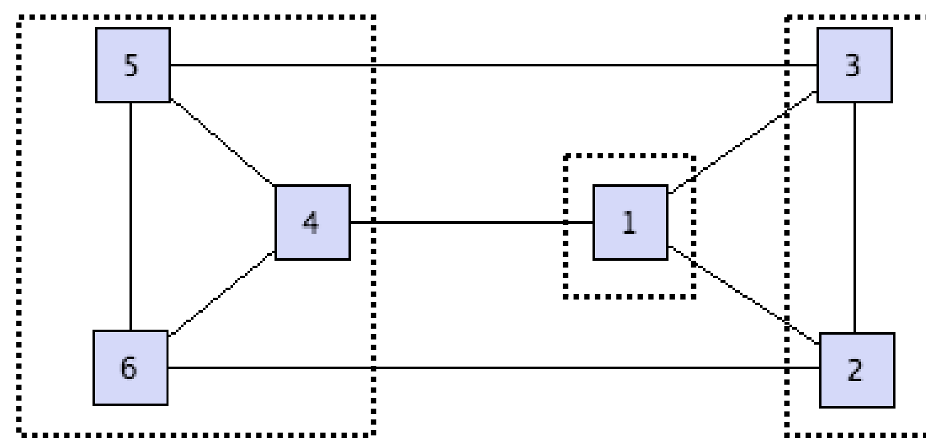
Q1



Q2



Q3



Q4

....

Modularity

Q = fraction of edges within communities - expected fraction of such edges

Let us attribute each node *i* to a community c_i

$$Q = \frac{1}{2m}\sum_{i,j}\Big[A_{ij} - P_{ij}\Big]\delta(c_i, c_j)$$

$$Q \in [-1, 1]$$

$$P_{ij} = \frac{k_i k_j}{2m}$$

expected number of links between i and j

$$\rightarrow \quad Q_C = \frac{1}{2m}\sum_{i,j}\Big[A_{ij} - k_i k_j/2m\Big]\delta(c_i, c_j)$$

Modularity optimisation

Different types of algorithm for different applications:

Small networks ($<10^2$): Simulated Annealing

Intermediate size (10^2 - 10^4): Spectral methods, PL, etc.

Large size ($>10^4$): greedy algorithms

Modularity

Resolution limit

Optimising modularity uncovers one partition

What about sub (or hyper)-communities in a hierarchical network?

Reichardt & Bornholdt

$$Q_\gamma = \frac{1}{2m} \sum_{i,j} \left[A_{ij} - \gamma P_{ij} \right] \delta(c_i, c_j)$$

Arenas et al.

$$Q(A_{ij} + r I_{ij})$$

Tuning parameters allow to uncover communities of different sizes

Reichardt & Bornholdt different of Arenas, except in the case of a regular graph where

$$\gamma = 1 + r / \langle k \rangle$$

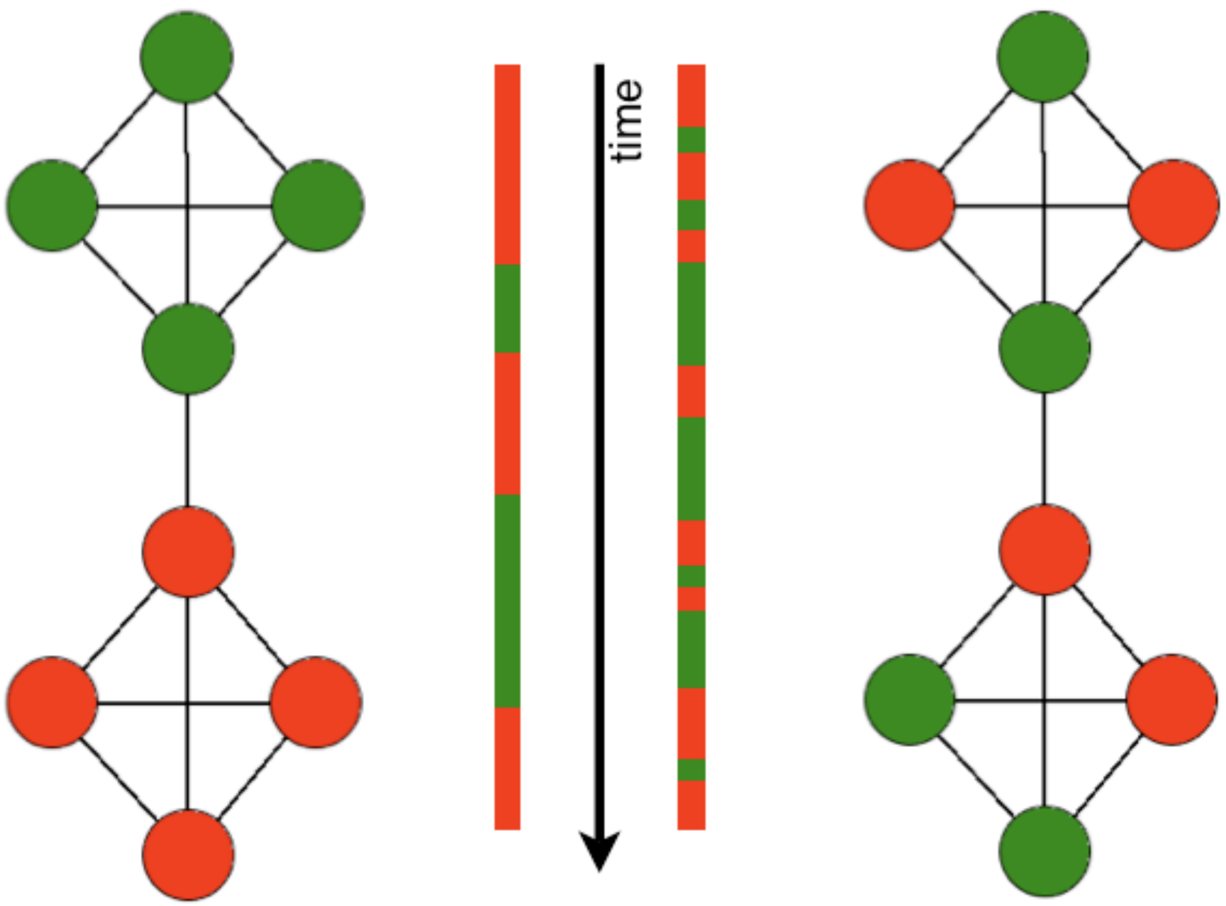
*J. Reichardt and S. Bornholdt, Phys. Rev. E **74**, 016110 (2006). Statistical mechanics of community detection*

*A Arenas, A Fernandez, S Gomez, New J. Phys. **10**, 053039 (2008). Analysis of the structure of complex networks at different resolution levels*

Stability

The quality of a partition is determined by the patterns of a flow within the network: a flow should be trapped for long time periods within a community before escaping it.

The stability of a partition is defined by the statistical properties of a random walker moving on the graph:



M. Rosvall and C. T. Bergstrom, PNAS 105, 1118 –1123 (2008)

J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. arXiv:0812.1811.

Stability

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$$R(t) = \sum_{C \in \mathcal{P}} P(C, t_0, t_0 + t) - P(C, t_0, \infty)$$

$P(C, t_0, t_0 + t)$ probability for a walker to be in the same community at times t_0 and $t_0 + t$ when the system is at equilibrium

$P(C, t_0, \infty)$ probability for two independent walkers to be in C (ergodicity)

J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. arXiv:0812.1811.

a. Modularity vs Stability

Let us consider a random walk on an undirected network:

$$p_{i;n+1} = \sum_j \frac{A_{ij}}{k_j} p_{j;n}$$

→ equilibrium

$$p_i^* = k_i/2m$$

$$R(1) = \sum_{i,j} \left[\frac{A_{ij}}{k_j} \frac{k_j}{2m} - \frac{k_i k_j}{(2m)^2} \right] \delta(c_i, c_j)$$

Probability that a walker is in the same community initially and at time t=1

Same probability for independent walkers

$$R(1) = Q = \frac{1}{2m} \sum_{i,j} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j)$$

a. Modularity vs Stability

Let us consider a random walk on a directed network:

$$p_{i;n+1} = \sum_j \frac{A_{ij}}{k_j^{out}} p_{j;n} \quad \xrightarrow{\text{equilibrium}} \quad p_i^* = \pi_i$$

$$R(1) = \sum_{i,j} \left[\frac{A_{ij}}{k_j^{out}} \pi_j - \pi_i \pi_j \right] \delta(c_i, c_j) \neq Q$$

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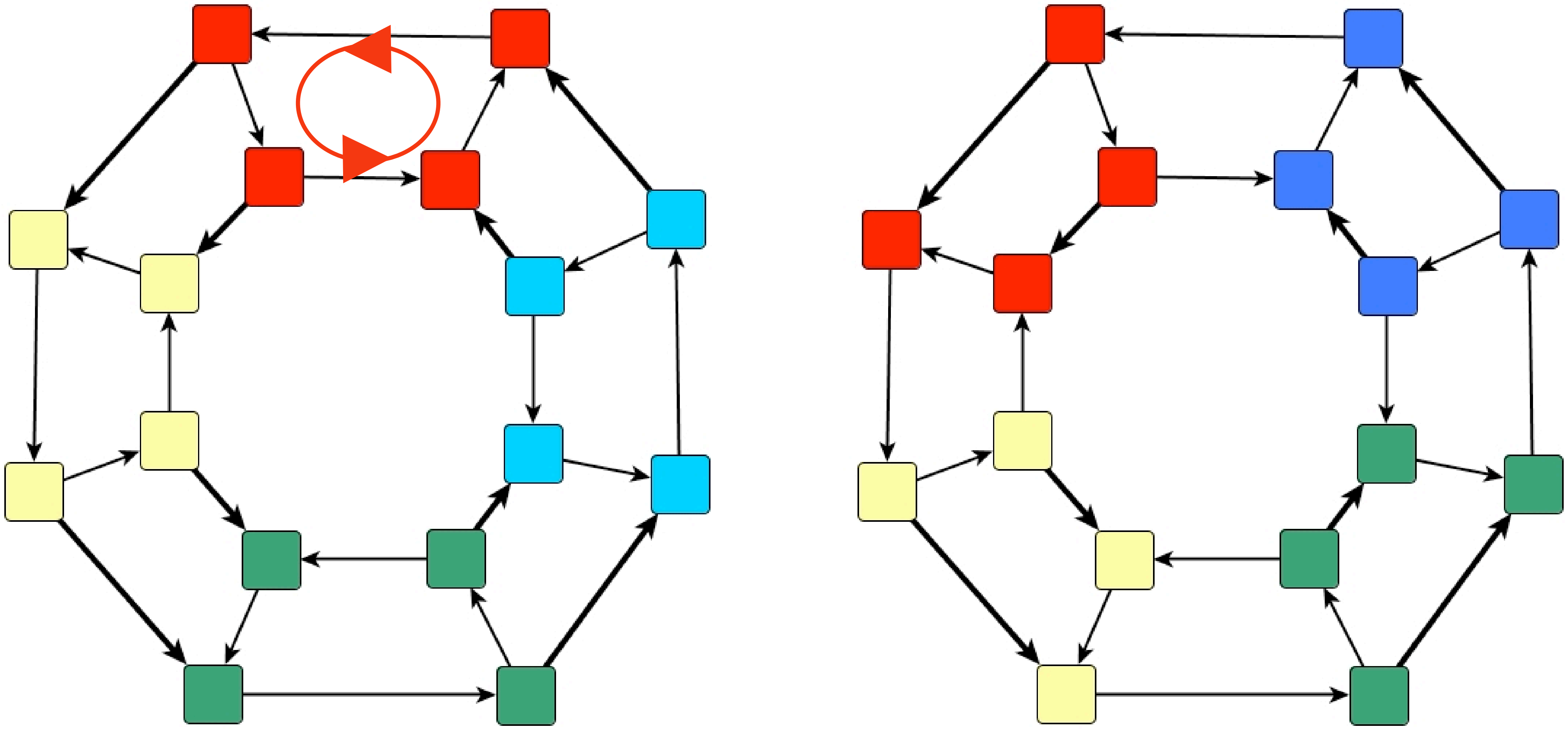

Flow-based modules | Combinatorial modules

M. Rosvall and C. T. Bergstrom, PNAS 105, 1118 –1123 (2008)

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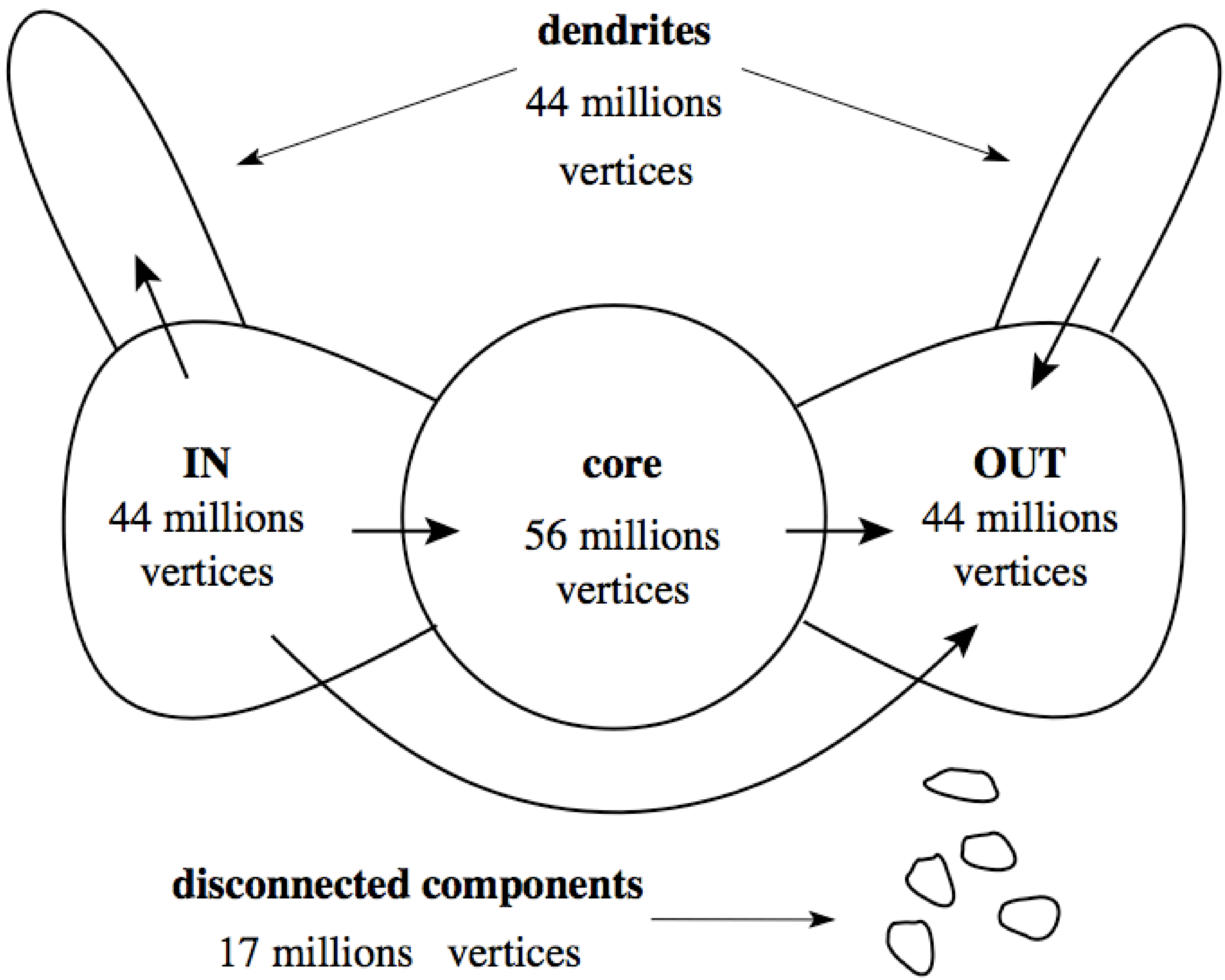

J.-L. Guillaume and M. Latapy, Proceedings of ALGOTEL'02

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$R(1) \neq Q(A)$ but $R(1) = Q(Y)$

$$Y = \frac{X + X^T}{2} \qquad X_{ij} = \frac{A_{ij}}{k_j^{out}} \pi_j$$

b. Stability: time as a resolution parameter

Let us consider a continuous-time random walk with Poisson waiting times

$$\dot{p}_i = \sum_j \frac{A_{ij}}{k_j} p_j - p_i \quad \xrightarrow{\text{equilibrium}} \quad p_i^* = k_i/2m$$

$$R(t) = \sum_{i,j} \left[\left(e^{t(B-I)}\right)_{ij} \frac{k_j}{2m} - \frac{k_i k_j}{(2m)^2} \right] \delta(c_i, c_j)$$

$$B_{ij} = A_{ij}/k_j$$

Probability that a walker is in the same community initially and at time t

Same probability for independent walkers

$$L_{ij} = A_{ij}/k_j - \delta_{ij}$$

Spectral decomposition: $$\frac{1}{2m} \sum_C \sum_{i,j \in C} \sum_{\alpha=2}^{N} e^{t\lambda_\alpha} v_{\alpha;i} v_{\alpha;j}$$

b. Stability: time as a resolution parameter

What are the optimal partitions of R_t?

t=0 $$R(0) = 1 - \sum_{i,j} \frac{k_i k_j}{(2m)^2} \delta(c_i, c_j)$$ → Communities=single nodes

t small $$R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$$

$(1-t)R(0)$ → favours single nodes; tQ_C → modularity

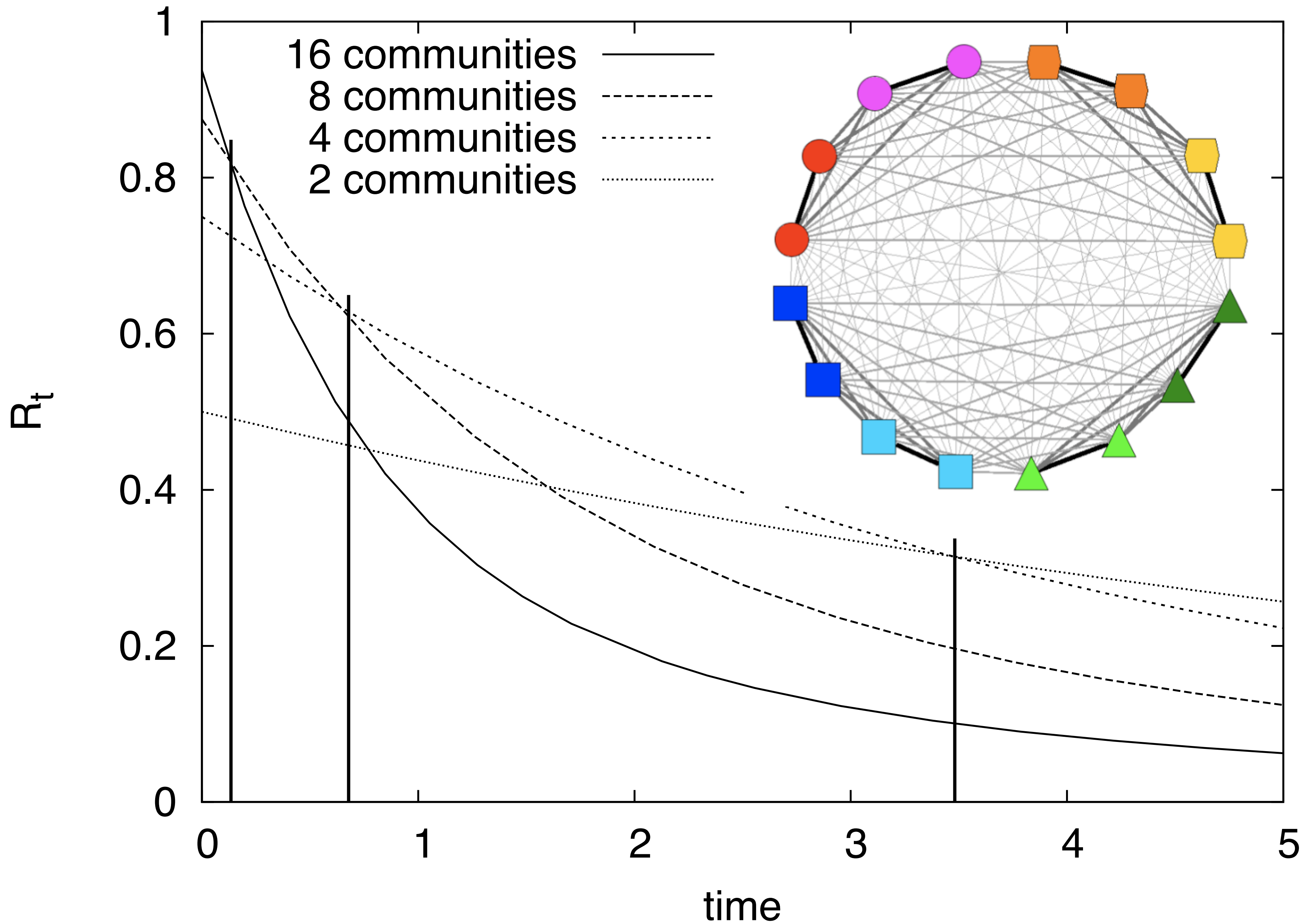
!! Q_t equivalent to the Hamiltonian formulation of Reichardt and Bornholdt (t=1/gamma)

....

When t goes to infinity, the optimal partition is made of 2 communities (by spectral decomposition, i.e. dominant eigenvector of the Laplacian)

b. Stability: time as a resolution parameter

Time is a “resolution parameter”: larger and larger communities when time is increased



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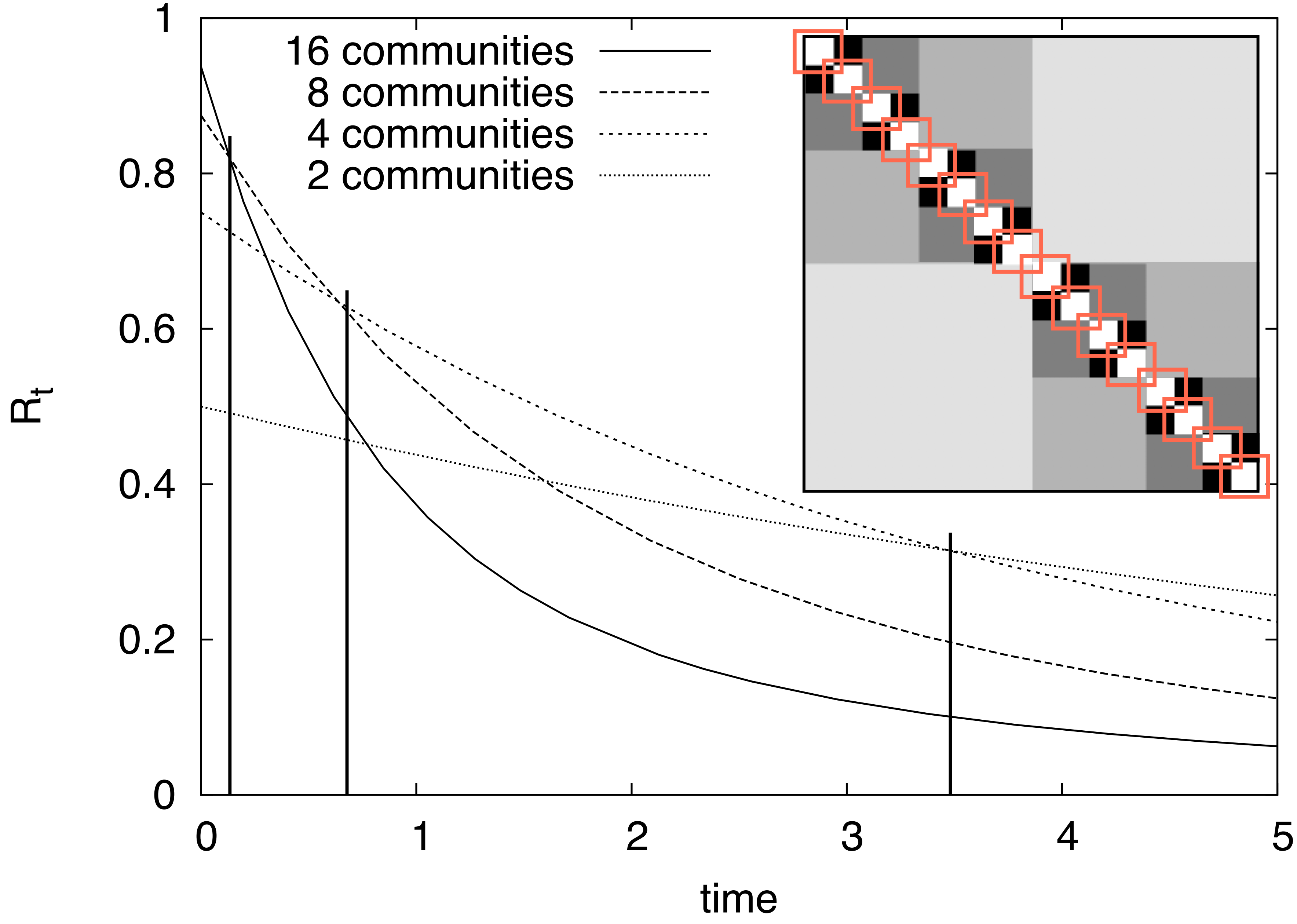

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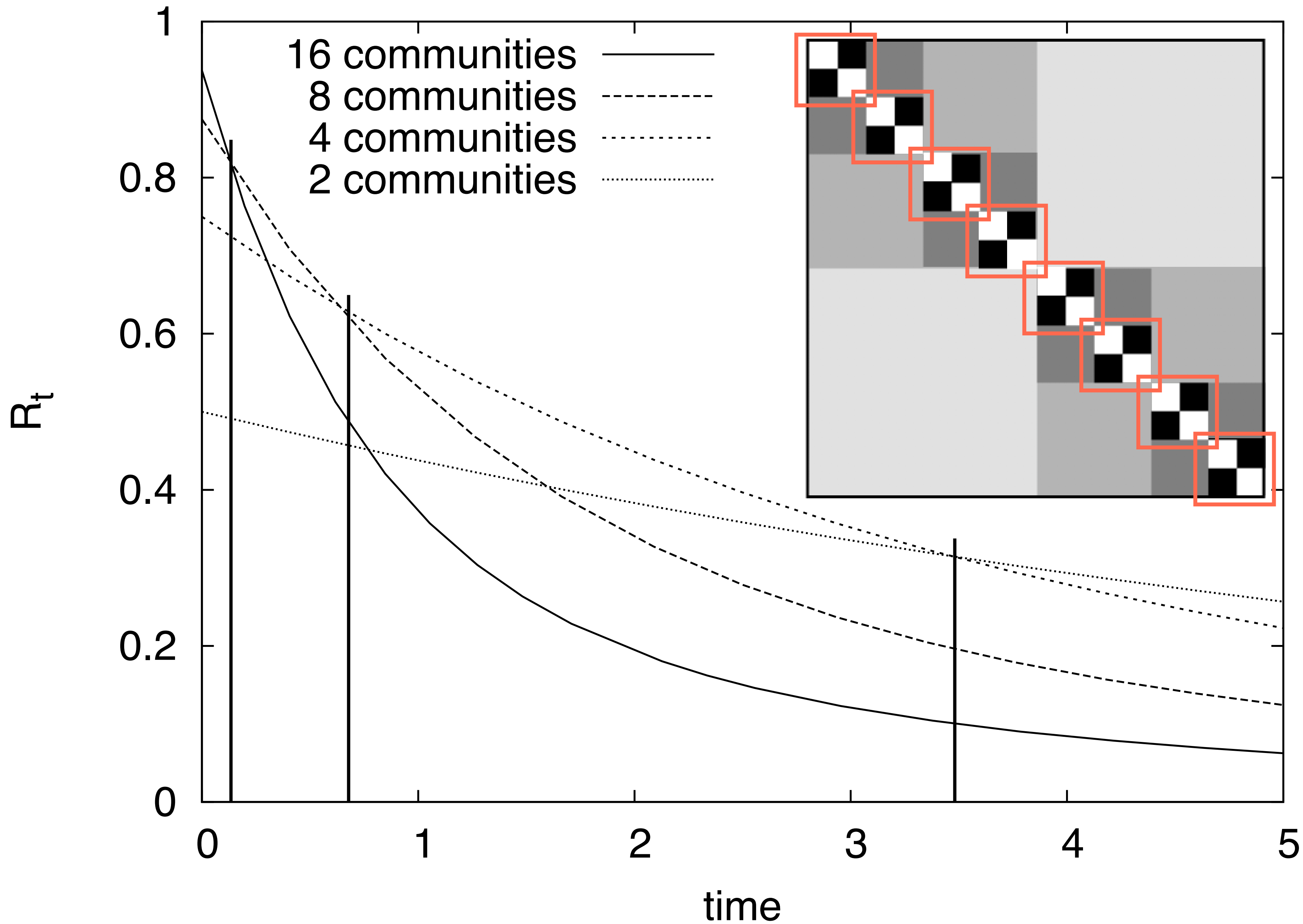

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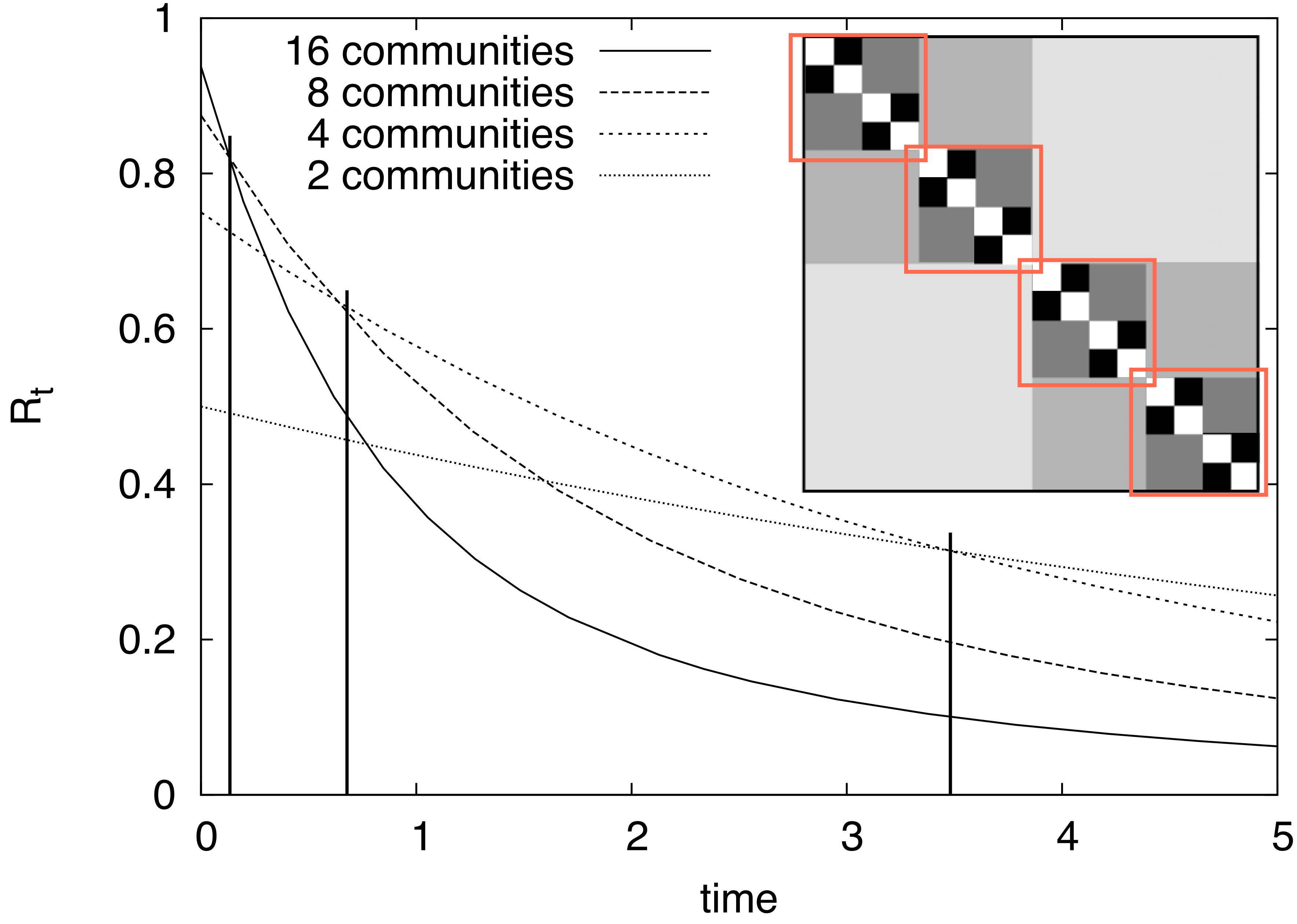

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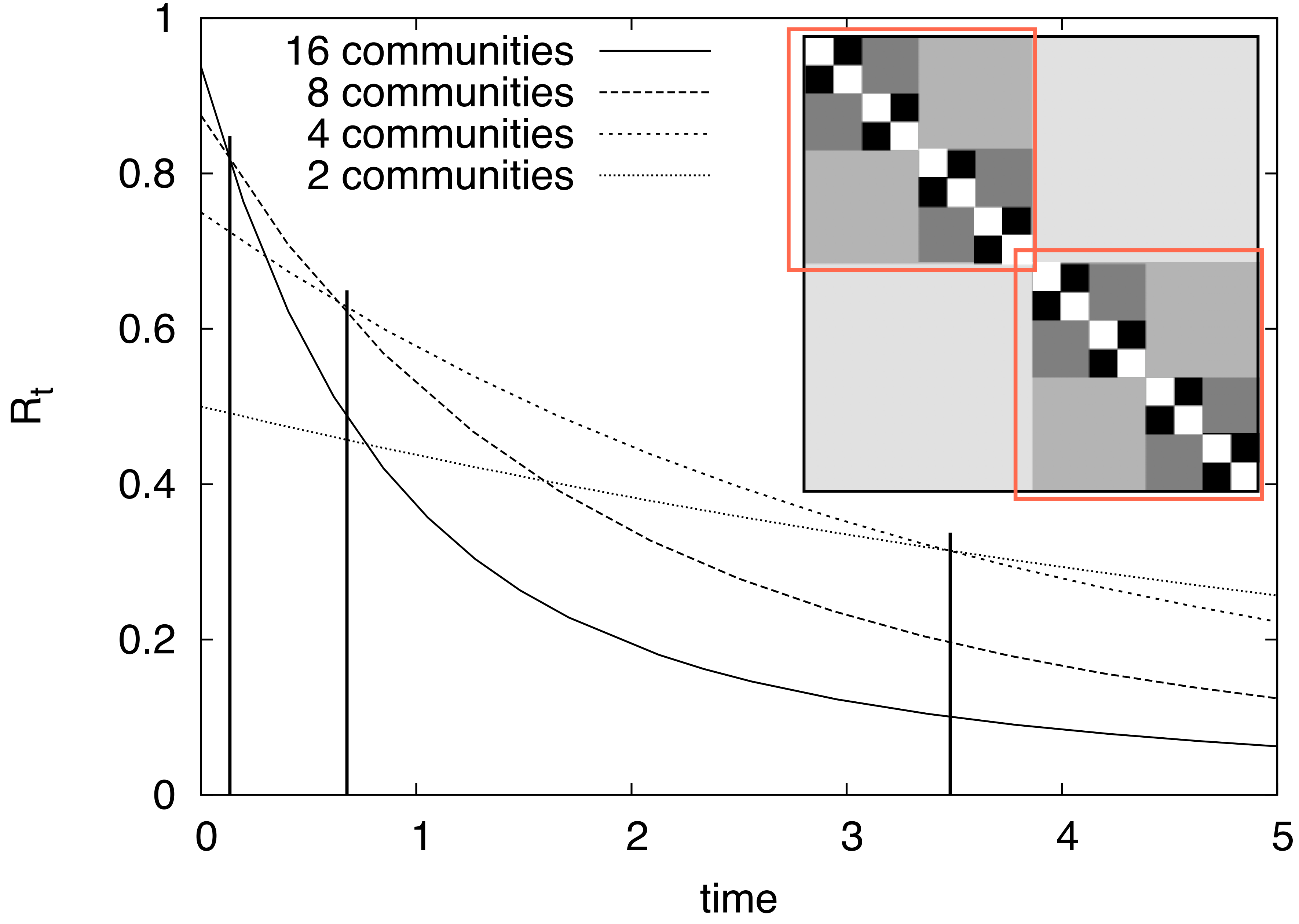


In practice: Optimisation

The stability *R(t)* of the partition of a graph with adjacency matrix *A* is equivalent to the modularity *Q* of a time-dependent graph with adjacency matrix *X(t)*

$$X_{ij}(t) = \left(e^{t(B-I)}\right)_{ij} k_j \qquad X_{ij}(t) = X_{ji}(t)$$

which is the flux of probability between 2 nodes at equilibrium and whose generalised degree is

$$\sum_j X_{ij}(t) = k_i$$

$$R(t) = \sum_{i,j} X_{ij}(t)/2m - k_i k_j/(2m)^2 \ \delta(c_i, c_j) = Q(X(t))$$

For very large networks: $R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$

In practice: Selection of the most relevant scales

The optimization of R(t) over a period of time leads to a sequence of partitions that are optimal at different time scales.
How to select the most relevant scales of description?

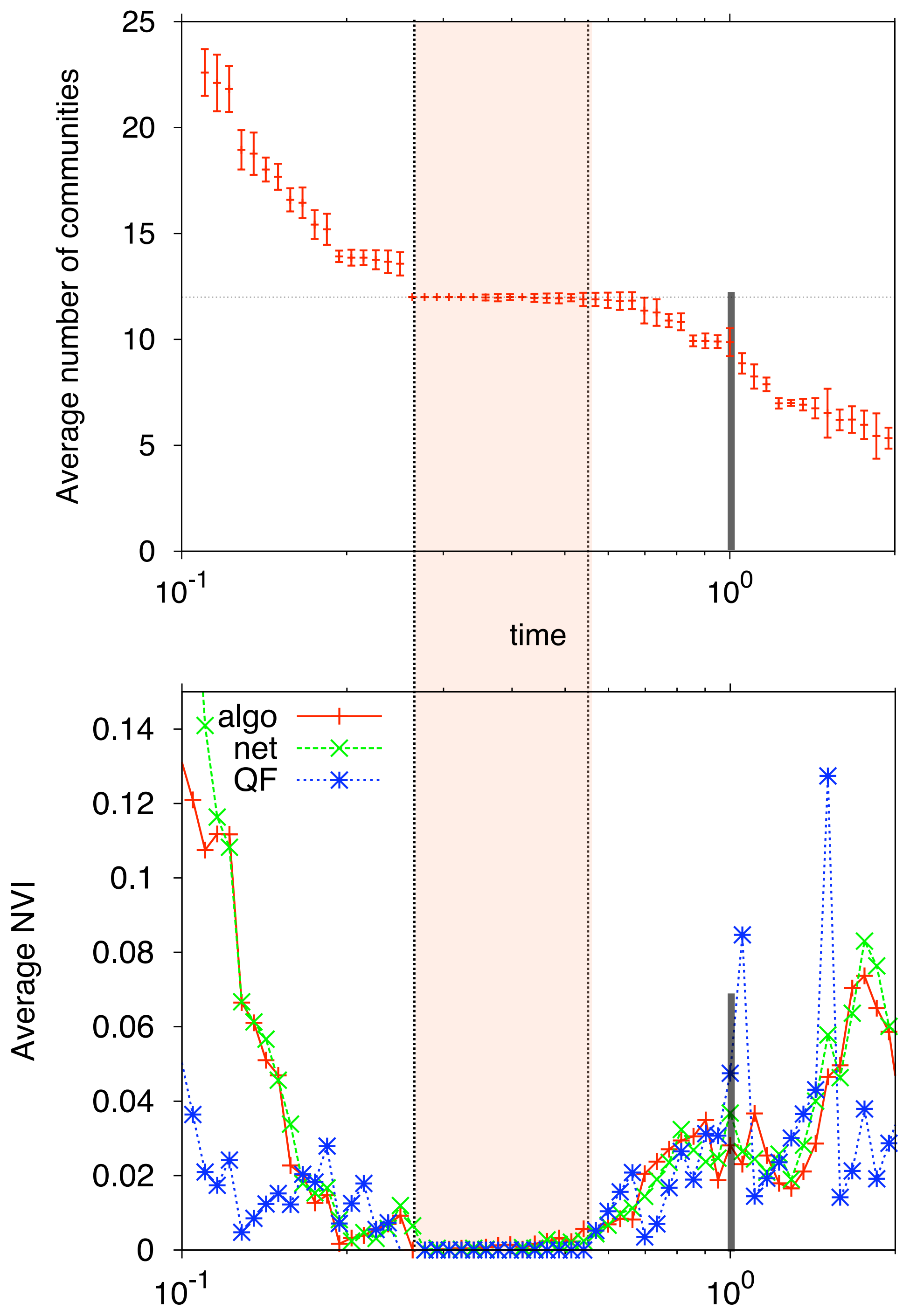
The significance of a particular scale is usually associated to a certain notion of the robustness of the optimal partition. Here, robustness indicates that a small modification of the optimization algorithm, of the network, or of the quality function does not alter this partition.

We look for regions of time where the optimal partitions are very similar. The similarity between two partition is measured by the normalised variation of information.

D.J. Fenn, M.A. Porter, M. McDonald, S. Williams, N.F. Johnson, N.S. Jones, *Chaos* 19, 033119 (2009).
M. Meilă, *J. Multivariate Anal.* 98, 873 (2007).

Verification on empirical networks and multi-scale benchmarks

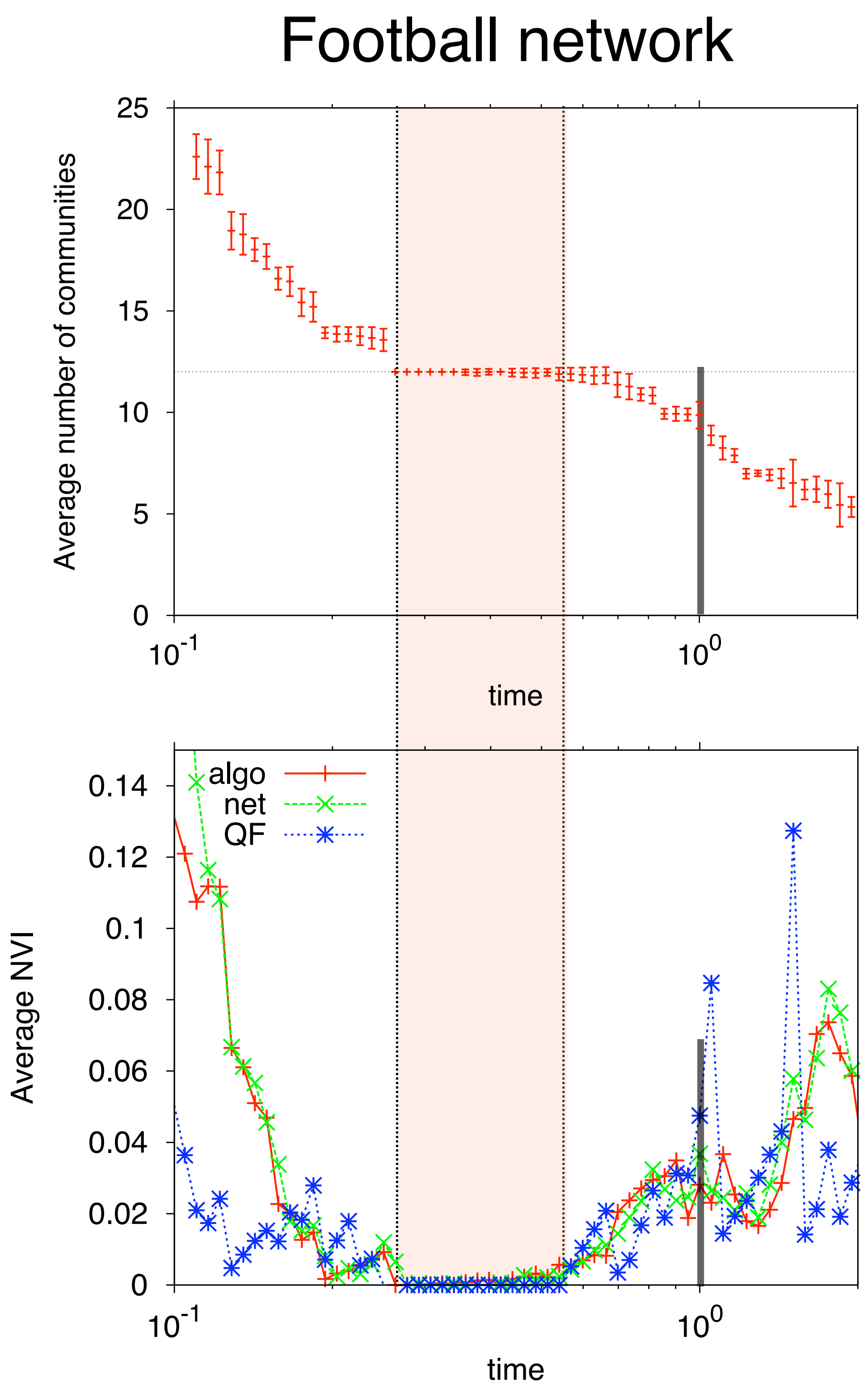
Football network



algo: for each t, 100 optimizations of Louvain algorithm while changing the ordering of the nodes

$$\langle V\rangle_{\mathrm{algo}}(t) = \frac{2}{T(T-1)} \sum_{i=1}^{T} \sum_{i'=i+1}^{T} \hat{V}(\mathcal{P}_i(t), \mathcal{P}_{i'}(t)),$$

Verification on empirical networks and multi-scale benchmarks



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net: for each t, 100 optimizations with a fixed algorithm but randomized modifications of the network

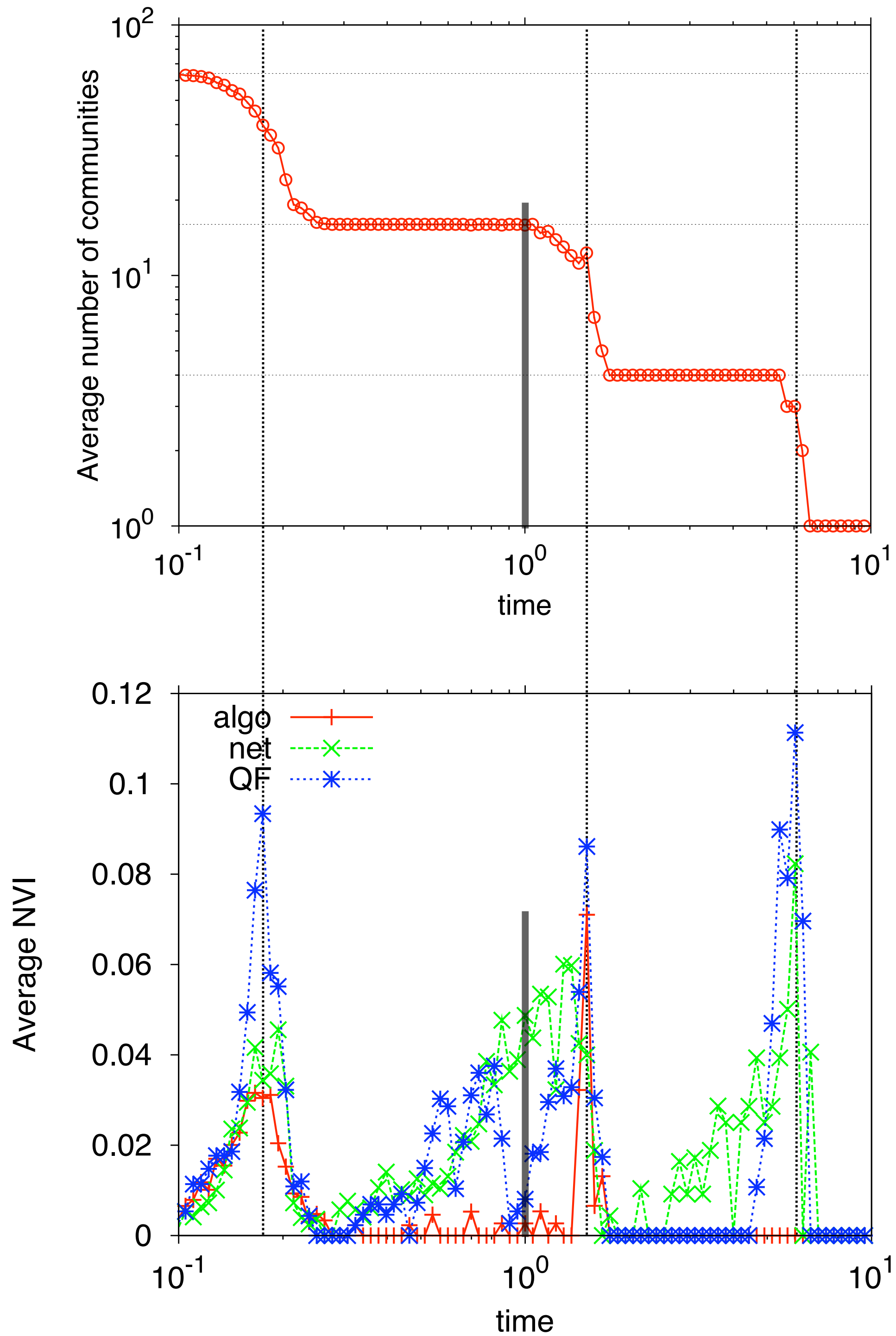
QF: for each t, one optimization. Partitions at 5 successive values of t are compared.

Compatible notions of robustness

Lack of robustness => high degeneracy of the QF landscape: uncovered partitions are not to be trusted; wrong resolution

Verification on empirical networks and multi-scale benchmarks

Multi-scale network



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What about dynamical networks?

Networks are intrinsic dynamical nature and the fact that networks are not frozen entities but are made of nodes and edges evolving at different time scales.

Firm empirical basis: “Network science” is driven by the analysis of empirical data.

More and more large-scale data-sets are longitudinal

Need for the development of new methods in order to detect structural changes in large-scale networks.

Workshop on the
Analysis of Mobile Phone Networks

A satellite workshop to NetSci 2010
Tuesday, May 11, 2010
MIT, Cambridge, MA

11:00-11:15 Timescales in evolving mobile networks
G.M. Krings (1), M. Karsai (2), J. Saramäki (2), V.D. Blondel (1)
(1) UCLouvain
(2) BECS, School of Science and Technology, Aalto University

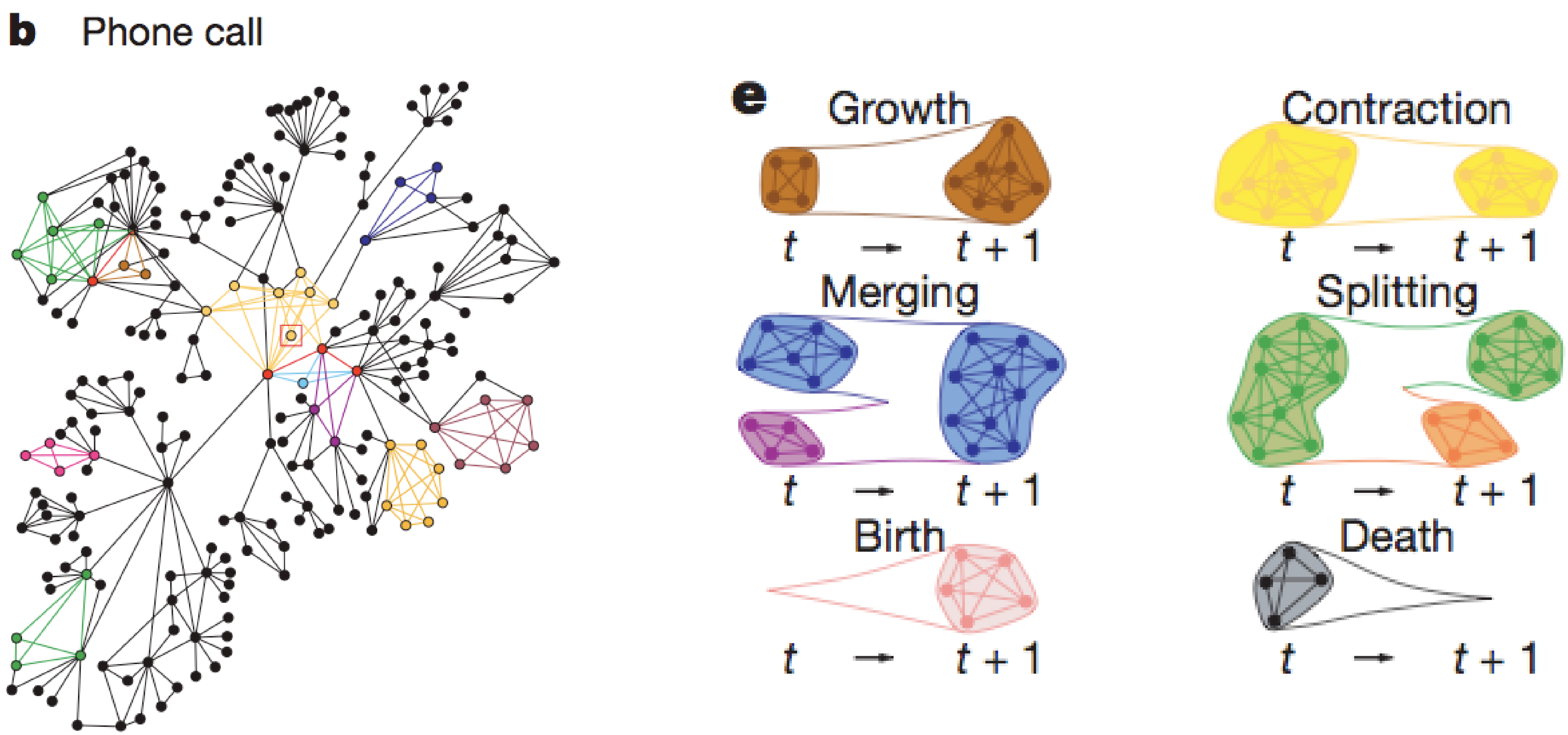
11:15-11:30 Towards an investigation of the structure and temporal dynamics in a large scale telecoms dataset
F. Reid, N. Hurley
Clique Research Cluster, University College Dublin

11:30-11:45 Dynamics and temporal correlations in mobile phone based social networks
M. Karsai (1), L. Kovanen (1), M. Kivelä (1), R. K. Pan (1), J. Saramäki (1), J. Kertész (2), A.-L. Barabási (3,4), K. Kaski (1)
(1) BECS, School of Science and Technology, Aalto University
(2) Institute of Physics, Budapest University of Technology and Economics
(3) CCNR, Northeastern University
(4) CCSB, Dana-Farber Cancer Institute

What about dynamical networks?

Simplest approach:

Compare the modules obtained at different snapshots



G. Palla, A.-L. Barabási, T. Vicsek, Nature 446:7136, 664-667 (2007)

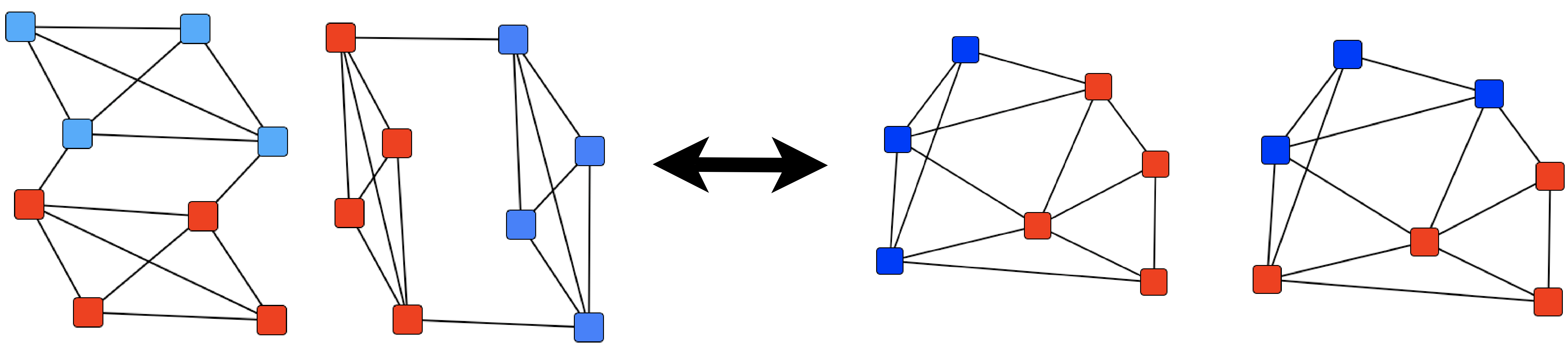
What about dynamical networks?

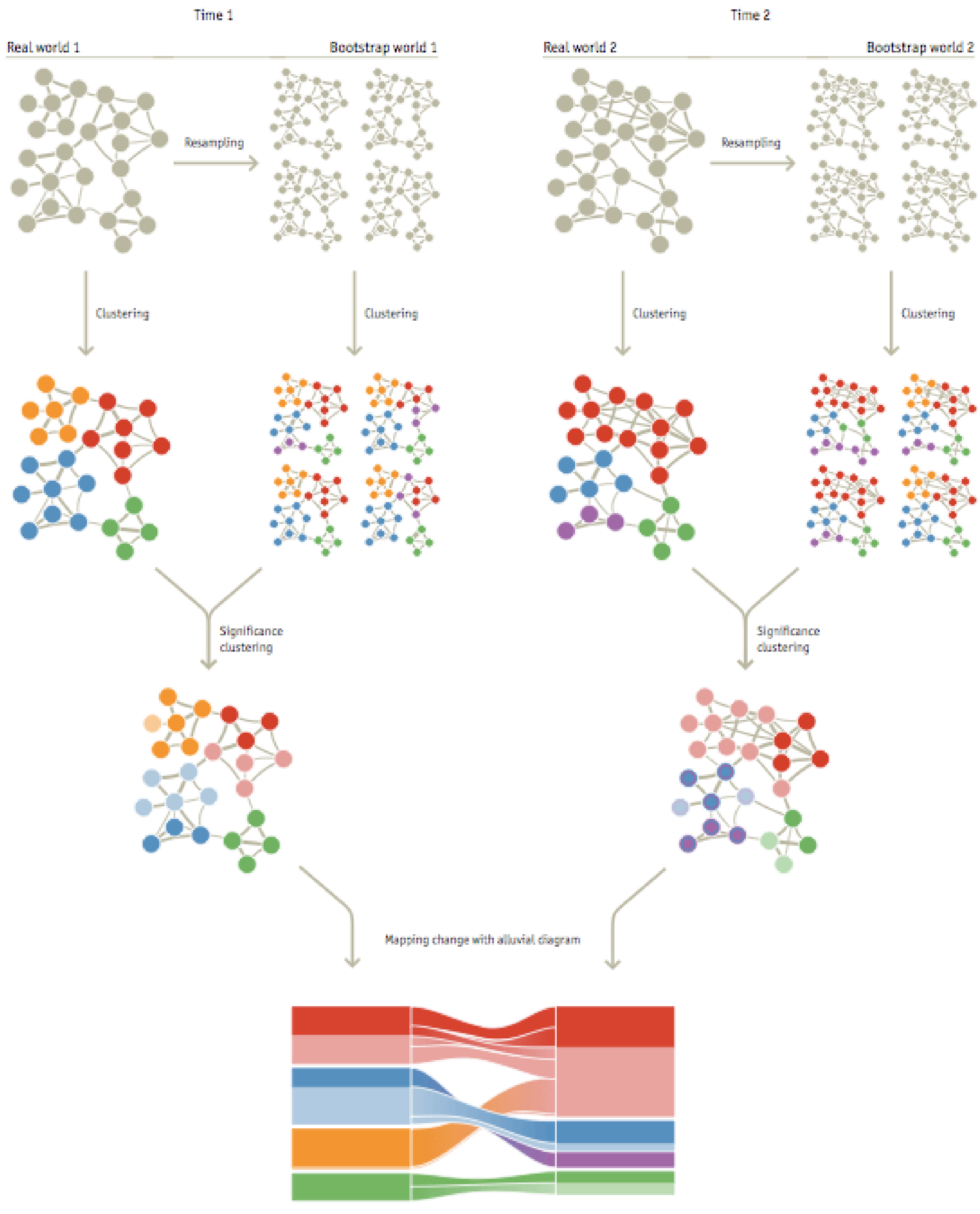
But blind to the difference between noise and trend in the data.

One first needs to determine the significance of the partition of single networks in order to distinguish structural changes from random fluctuations (“degeneracy”, robustness, etc.) and assess how much confidence we should have in the changes.

Do the clusters change more than they would because of a small random modification of the network?

Global re-organization? Lack of robustness/sensibility





1) Bootstrap resampling: clustering on modified versions of the network

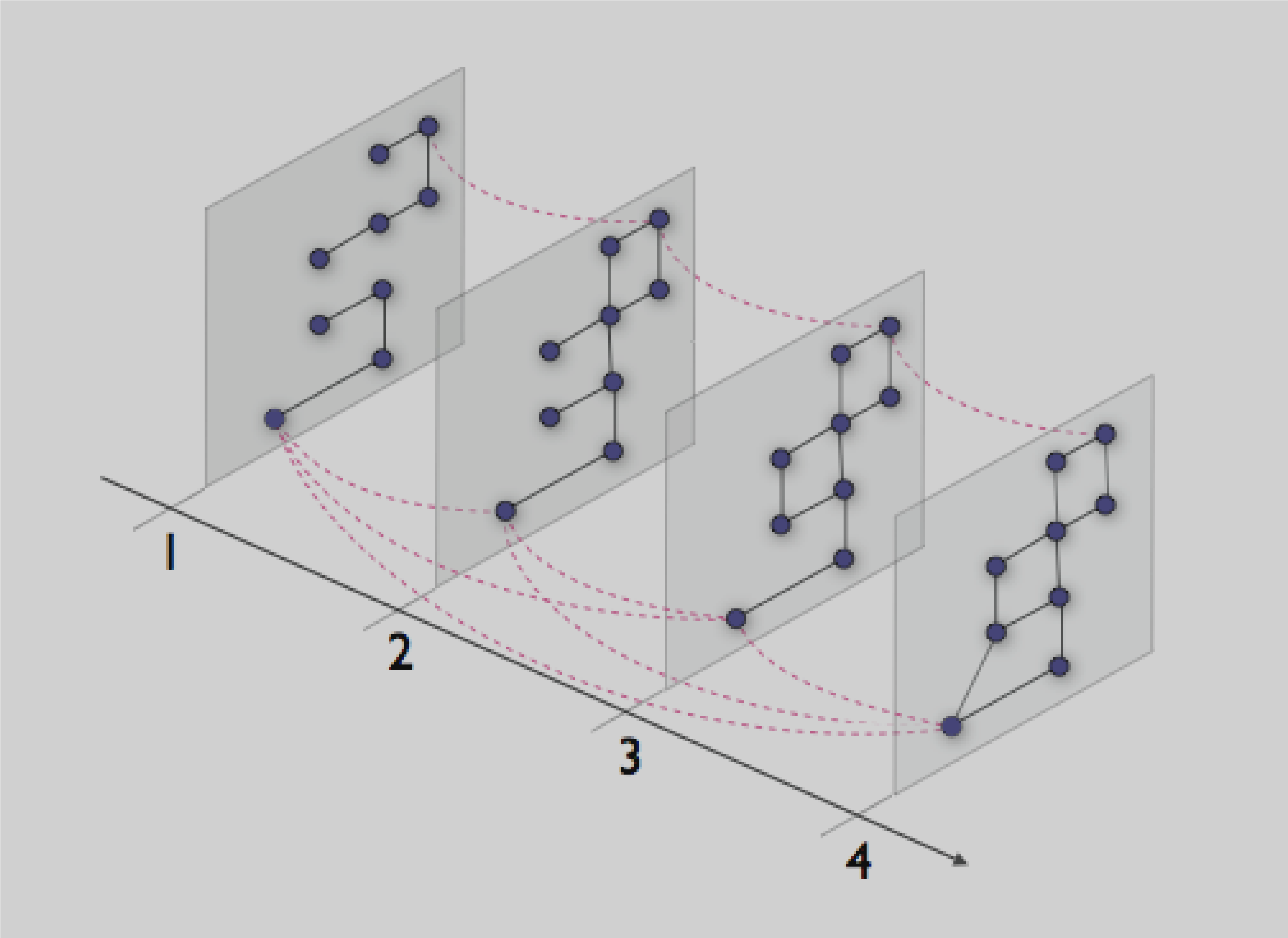
2) Significance clustering: modules that are present in a large number of instances

3) The procedure is repeated for networks at different values of time and significant clusters are compared (alluvial representation).

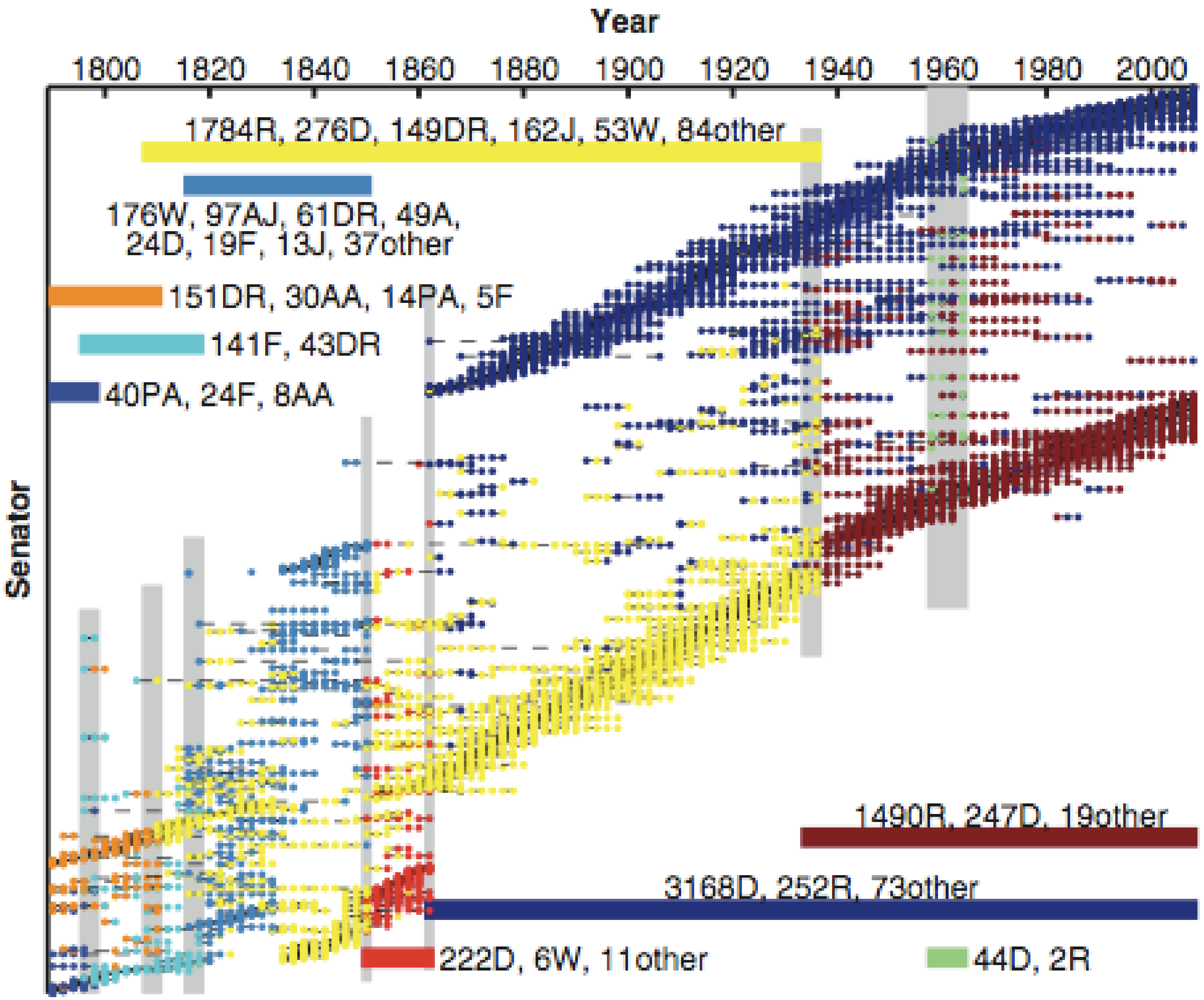
Mapping Change in Large Networks, M. Rosvall, C.T. Bergstrom PLoS One 2010 | Volume 5 e8694

What about dynamical networks?

“Multi-slice networks”: coupling between same nodes in different instances of the networks (“inertia”). Partitioning methods directly on multi-slice networks. Links between slices artificially stabilize the partitions



Mucha, Peter J.; Richardson, Thomas; Macon, Kevin; Porter, Mason A.; and Onnela, Jukka-Pekka [2010]. Community Structure in Time-Dependent, Multi-scale, and Multiplex Networks, Science, Vol. 328, No. 5980: 876-878.



110 networks (one for each year): link between senators if they have similar voting patterns

“Network-temporal” modules, extended over time, recovering democratic and republican parties

Conclusion

- Relation between dynamics and the hierarchical structure of networks
- Dynamical formulation for the quality of a partition
- Modularity and Stability are radically different in the case of directed networks
- Changing time allows to zoom in and out
- Notions of robustness to uncover significant scales
- Dynamical networks

Original Louvain method to optimise modularity available on *http://findcommunities.googlepages.com*

Generalized codes to optimise Q_t available on http://*www.lambiotte.be*

Thanks to J.-L. Guillaume (for providing his c++ code)

R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*

R. Lambiotte, *arXiv:1004.4268*

V.D. Blondel, J.-L. Guillaume, R. Lambiotte and E. Lefebvre, J. Stat. Mech., P10008 (2008).

T. Evans and R. Lambiotte, *Phys. Rev. E*, **80** (2009) 016105

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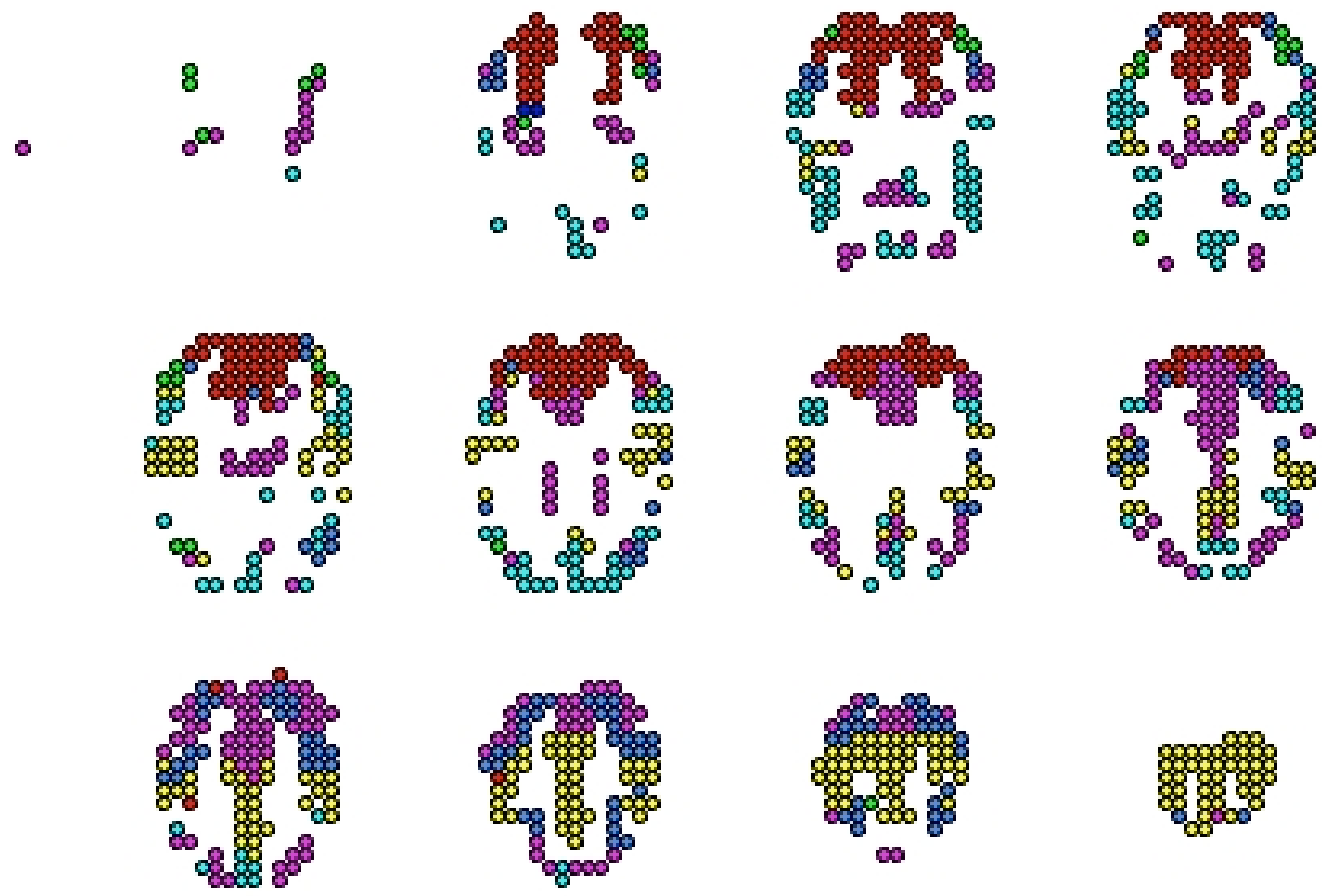
S. Yaliraki, K. Christensen, P. Expert, T. Evans, H. Jensen (Imperial College)
P. Panzarasa (Queen Mary)
D. Meunier (Cambridge)
V. Blondel (Louvain)



Hierarchical modularity in human brain functional networks, D. Meunier, R. Lambiotte, A. Fornito, K.D. Ersche, E.T. Bullmore, *Frontiers in NeuroInformatics* ***3****, 37 (2009)*