Opinion Dynamics on Heterogeneous Networks

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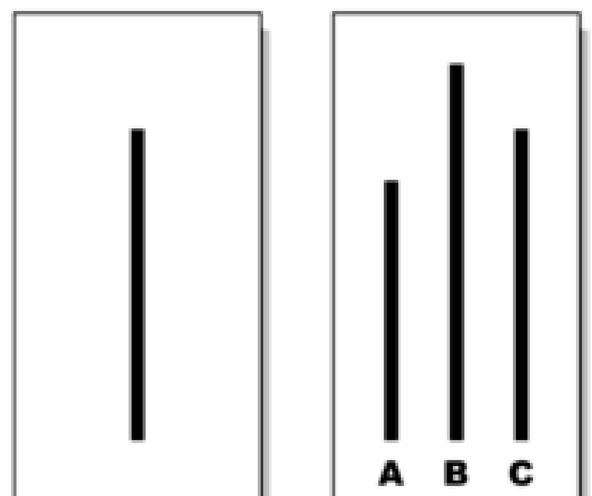
In collaboration with M. Ausloos,
S. Redner and J. Holyst

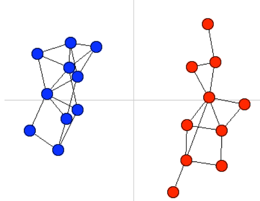
Contents

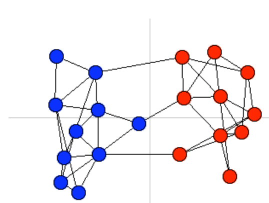
Opinion formation models, the “social atom”: Voter, Galam, Sznajd, etc.

Simplest interaction possible: imitation. People copy the behaviour of their friend, acquaintances, neighbours, etc.

E.g. Solomon Asch experiment:







1. Vacillating Voter Model

Dynamics of Vacillating Voters, R. Lambiotte and S. Redner, submitted to *JSTAT*, physics/0710.0914

Voter Model

N agents have an opinion: -1 or 1

The population evolves by:
(i) picking a random voter
(ii) the selected voter adopts the state of a randomly-chosen neighbor
(iii) repeating these steps *ad infinitum* or until a finite system necessarily reaches consensus.

The voter model is soluble in all dimensions

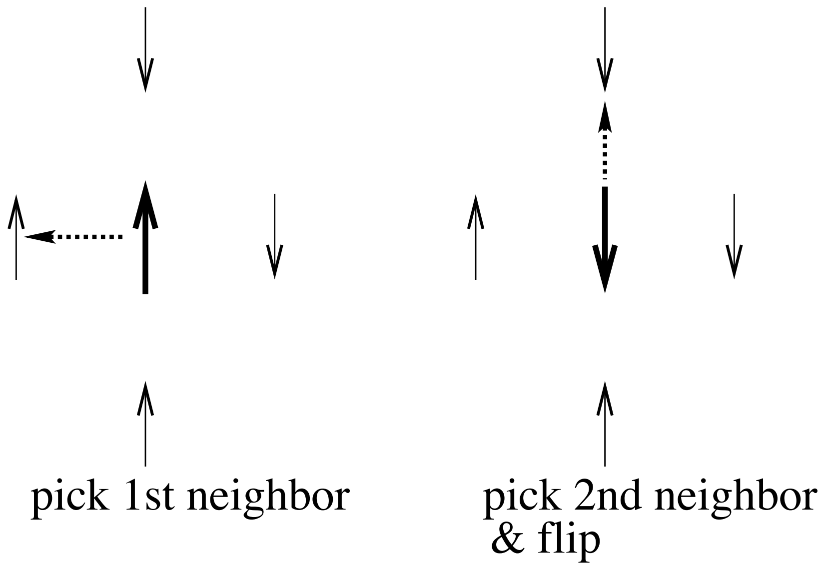
With this dynamics, a voter chooses a state with a probability equal to the fraction of neighbors in that state.
=> The average magnetization is conserved

Vacillating Voter Model

We investigate a variation that we term the vacillating voter model => the voters very much lack confidence in their state.

In an update, if a voter happens to select a random neighbor of the same persuasion, the voter is still not convinced that this state is right. Thus the voter selects another random neighbor and adopts this state.

This vacillation causes a voter to change state with a larger probability than the fraction of disagreeing neighbors.
This leads to a bias toward the zero-magnetization state in which there are equal densities of voters of each type (50-50).



Consequently, vacillation inhibits consensus.

But this is due to a different mechanism than that in the Axelrod model, the bounded compromise model and its variants. For these latter models, consensus is hindered because of the absence of interaction whenever two agents become sufficiently incompatible. For vacillating voters, it is individual uncertainty that inhibits consensus.

The vacillating voter model also differs from models that incorporate ”contrarians” because voters still try to imitate their neighbors.

Let us first consider first the mean-field limit.

$$\begin{aligned}\dot{x} &= -x\left[1-x^2\right]+(1-x)\left[1-(1-x)^2\right]\\ &= x(1-x)(1-2x).\end{aligned}$$

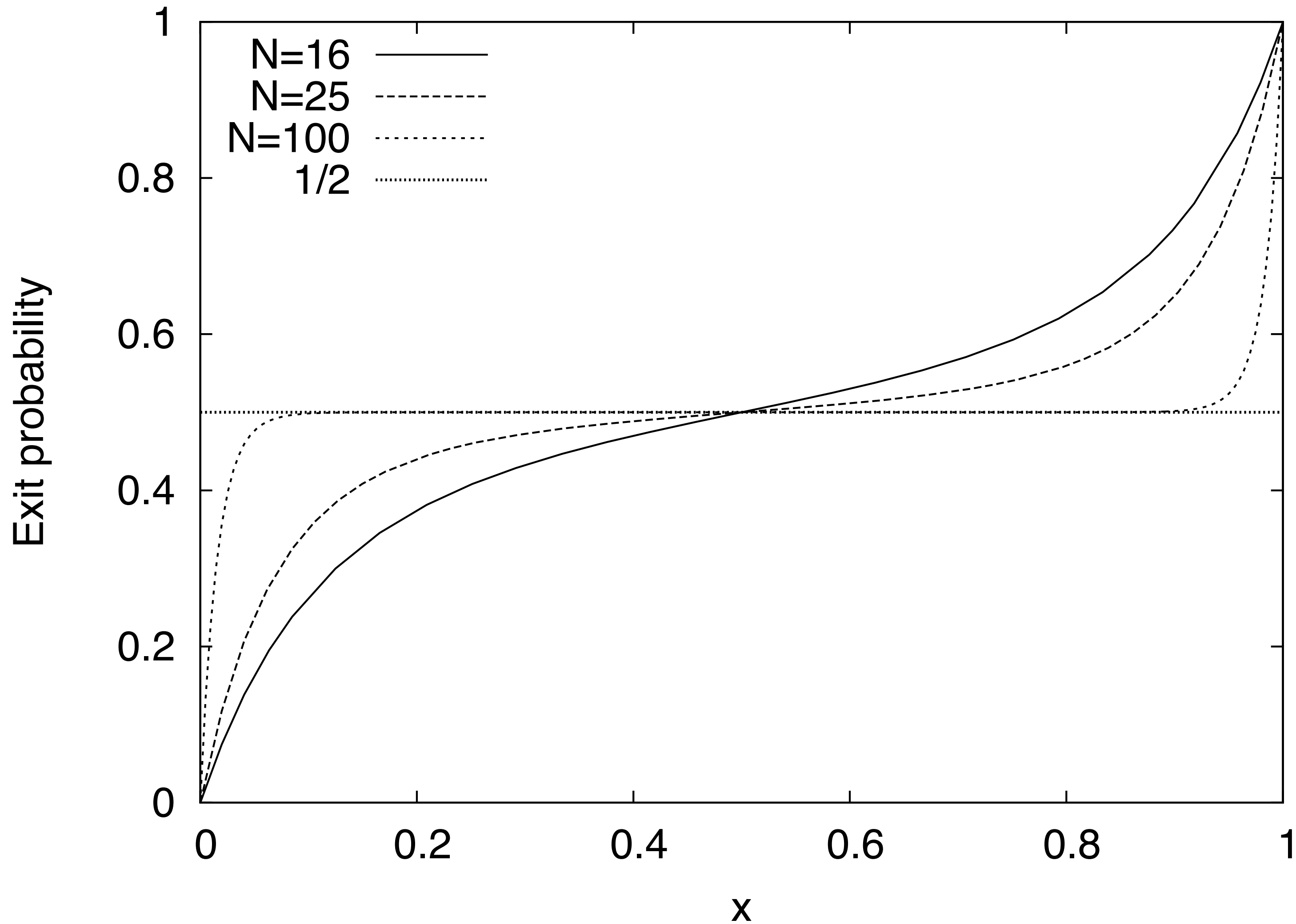
where x is the density of +1voters

The first term on the right accounts for the loss of +1 voters in which a +1 voter is first picked (factor *x*), and then the neighborhood does not consist of two +1 voters.

The factorized form shows that there are unstable fixed points at x=0, and a stable fixed point at x=1/2. Thus a population is driven to the zero-magnetization state.

Consensus is the only absorbing state of the stochastic dynamics => a finite population ultimately reaches consensus.
To characterize the evolution to this state, we focus on the exit probability $\mathcal{E}(x)$, defined as the probability that a population of N voters ultimately reaches +1 consensus when the initial fraction of +1 voters is x

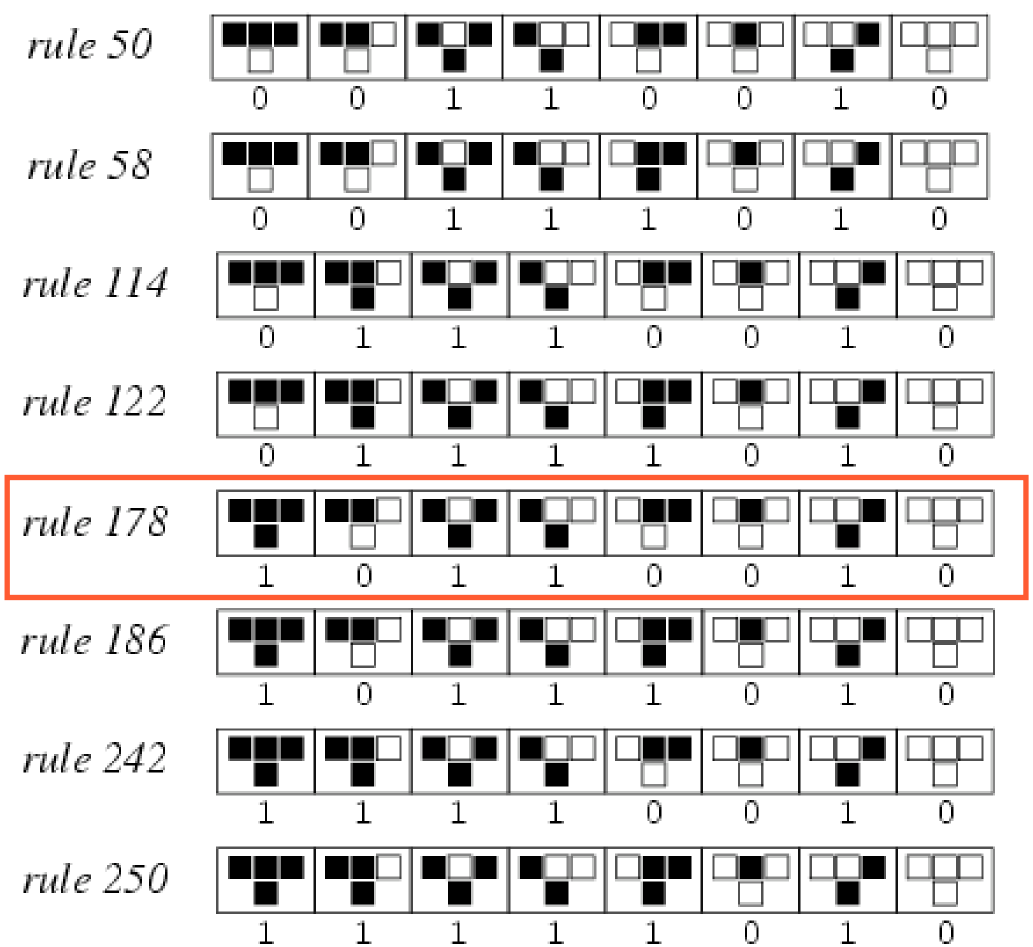
$$\mathcal{E}(x) = \int_{-1/2}^{x-1/2} e^{2Ny^2/3}\, dy \Big/ \int_{-1/2}^{1/2} e^{2Ny^2/3}\, dy$$



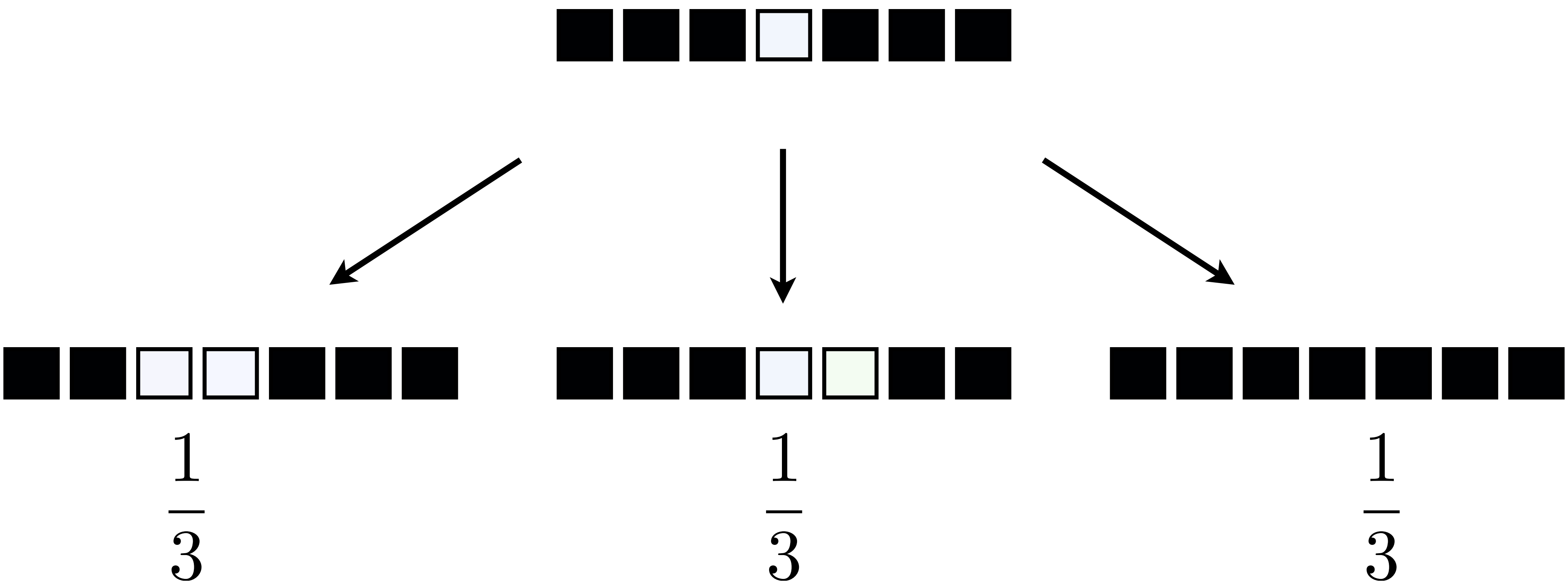
The exit probability approaches the constant value 1/2 for increasing N, reflecting the bias towards the zero-magnetization state. Almost all initial states are driven to the potential well at x=1/2, so that the exit probability becomes independent of the initial density of +1 voters.

In one dimension, a voter changes its opinion if at least one of its neighbors is in disagreement. For example, a +1 voter flips with rate 1 if at least one neighbour is -1.

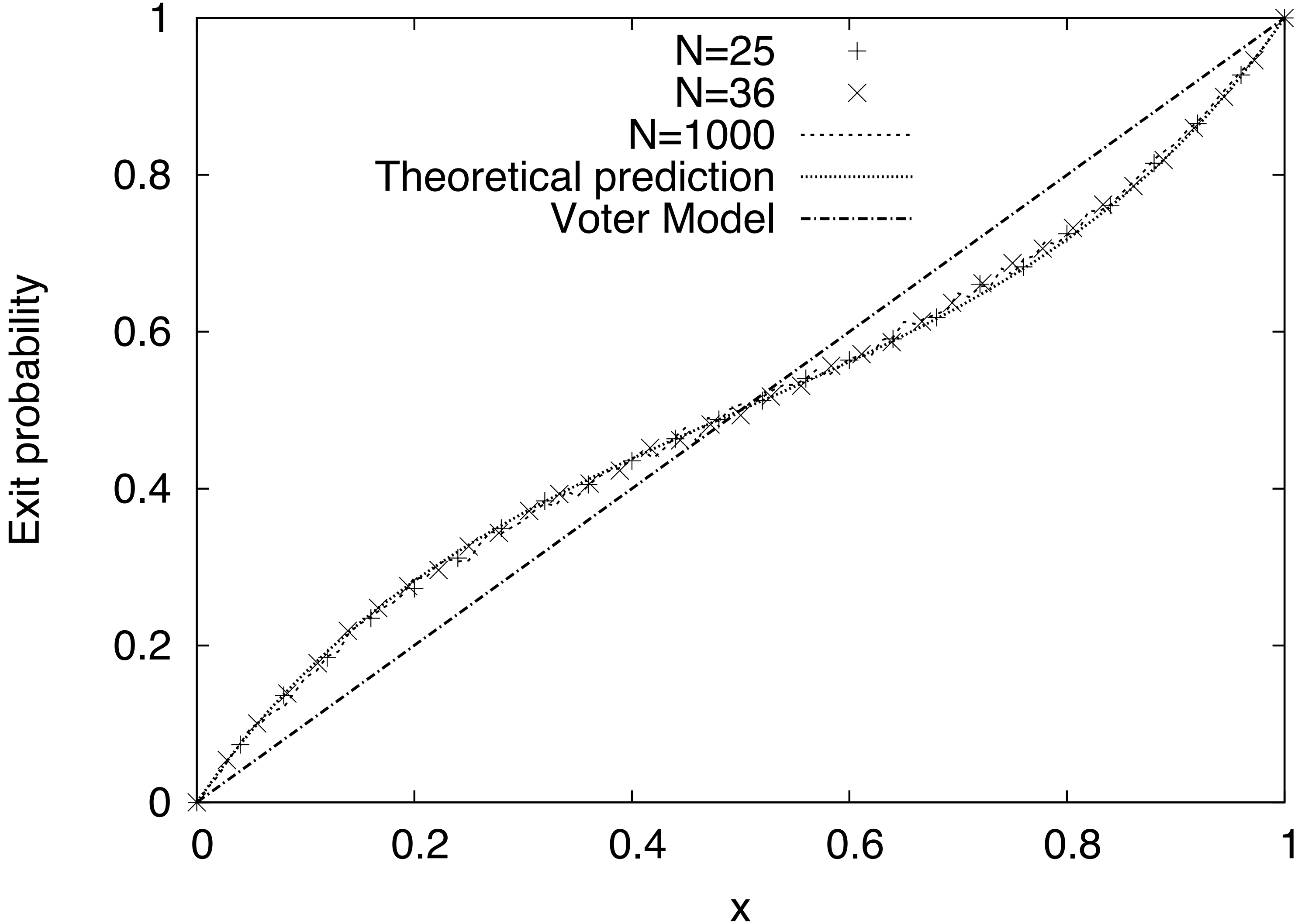
=> equivalent to rule 178 of the one-dimensional cellular automaton (see Wolfram), except that this rule is implemented asynchronously in the vacillating voter model.



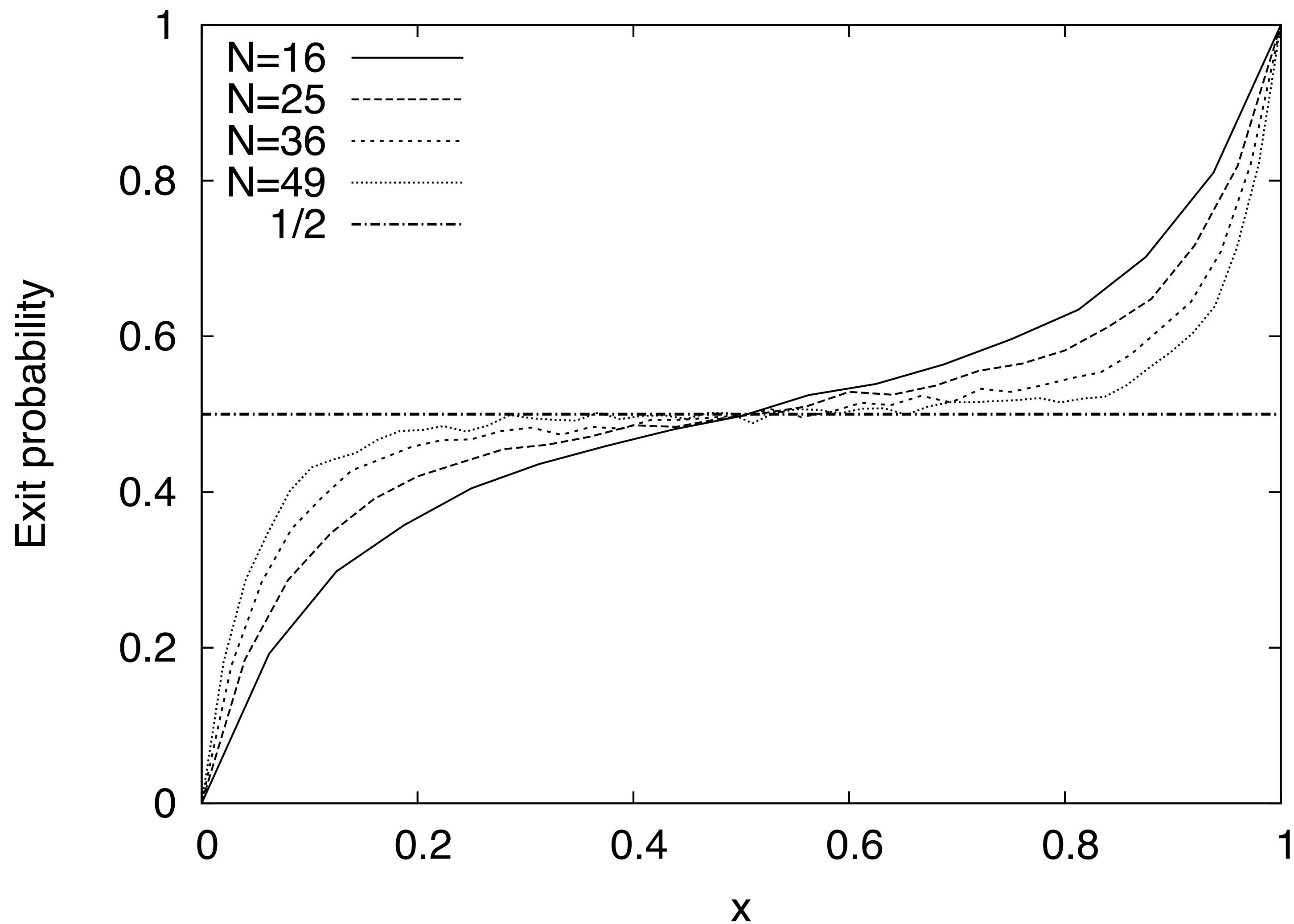
In one dimension, the system coarsens, albeit more slowly than in the pure voter model because of the repulsion of neighboring domain walls:



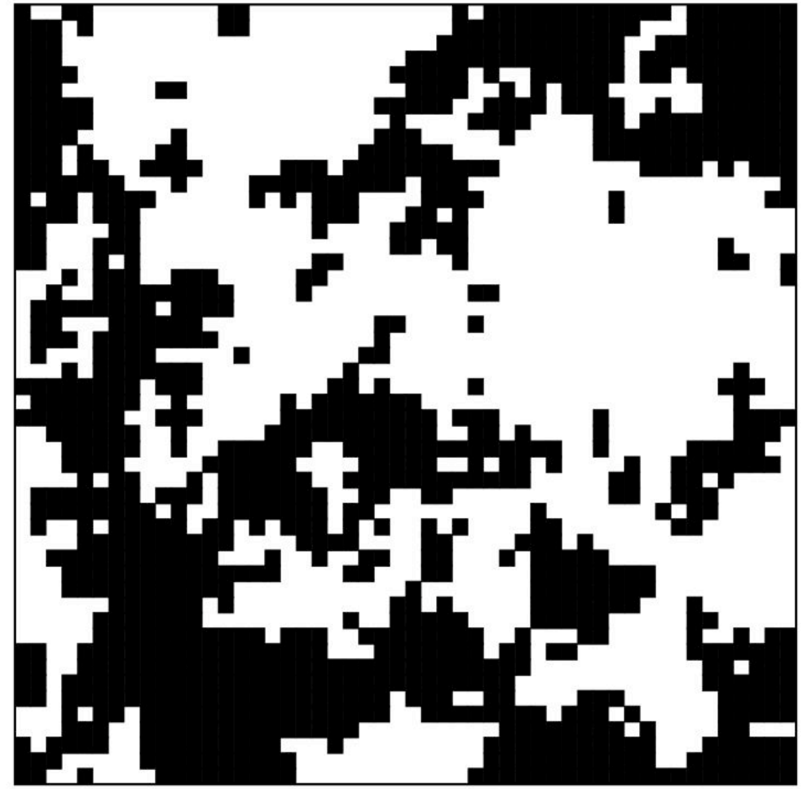
The probability to reach the final state of +1 consensus has a non-trivial initial state dependence.



In two and higher dimensions, small minority domains tend to grow => the system is driven to the zero-magnetization state. The overall behavior is qualitatively similar to that of the mean-field vacillating voter model, and very different from the pure voter model.



Vacillating Voter Model



Voter Model

$$C_1 \equiv \langle \sigma_{i,j}\sigma_{i,j+1} \rangle \rightarrow 0.31$$

→ domains of opposite opinions coexist

2. Majority Rule

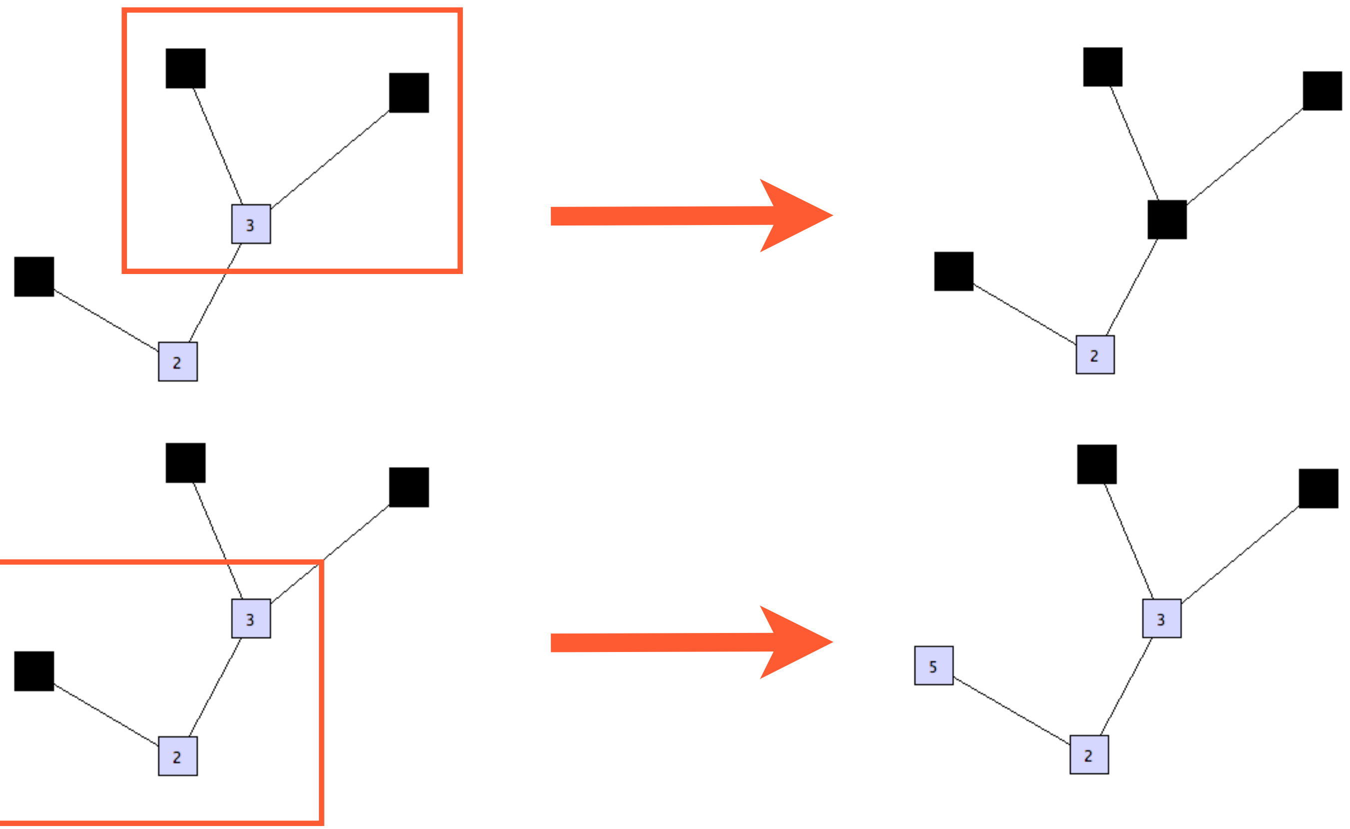
Coexistence of opposite opinions in a network with communities, R. Lambiotte and M. Ausloos, *JSTAT*, physics/0703266
Majority Model on a network with communities, R. Lambiotte, M. Ausloos and J. Holyst, *Phys. Rev. E,* 75 (2007) 030101(R)
How does degree heterogeneity affect an order-disorder transition?, R. Lambiotte, *EPL*, **78** (2007) 68002

Majority Rule

At each time step, one node is selected. Two processes:

i) With probability q, random change

ii) Else, a discussion between 2 of its neighbours takes place and the local majority is reached



Mean-field: Fully-Connected Network

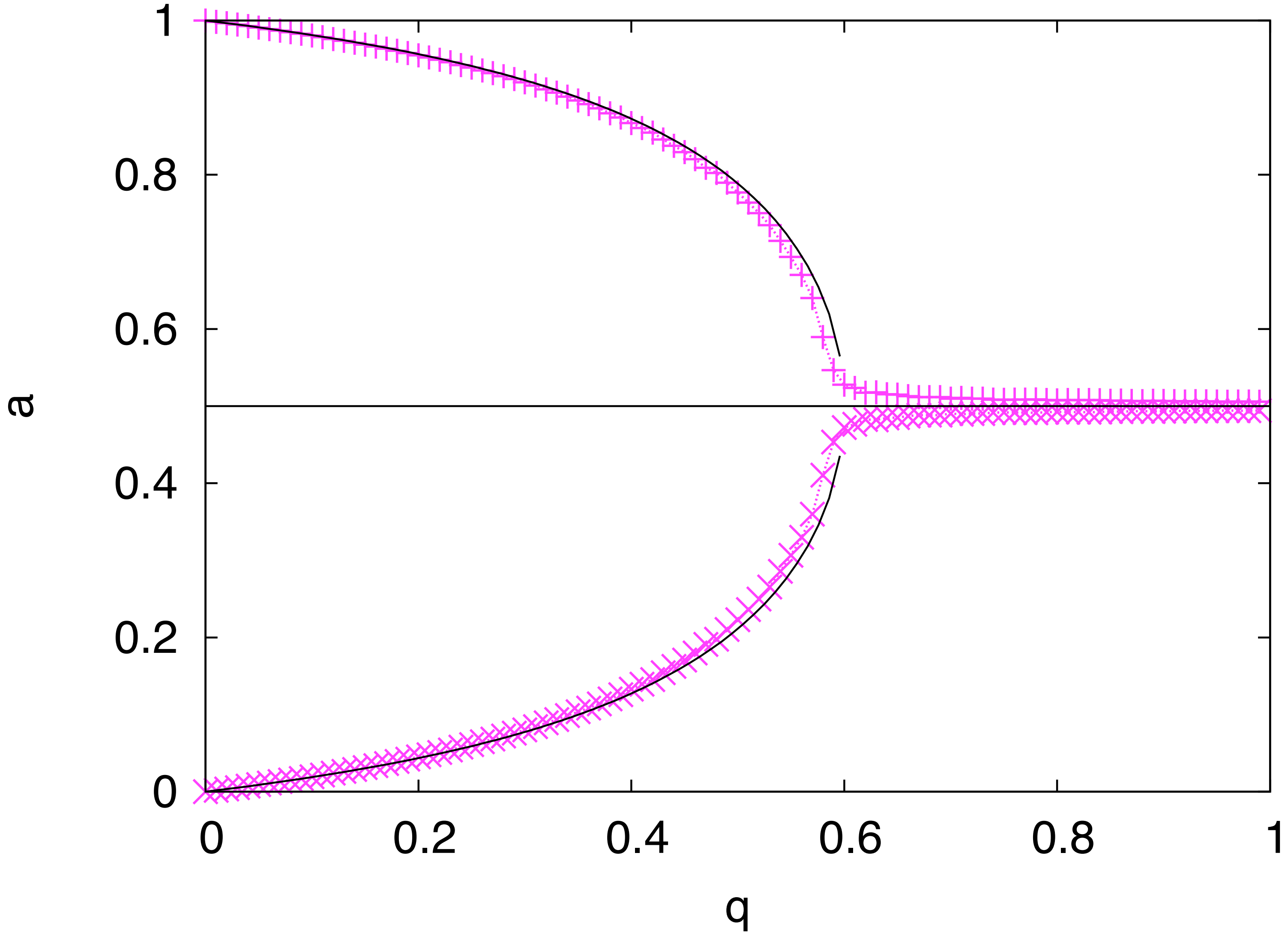
$$A_{t+1} = A_t + q(\frac{1}{2} - a_t) - 3(1-q)a_t(1 - 3a_t + 2a_t^2)$$

A_t average number of nodes with opinion α at time *t*

$a_t \equiv \frac{A_t}{N}$ average proportion of nodes with opinion α at time *t*

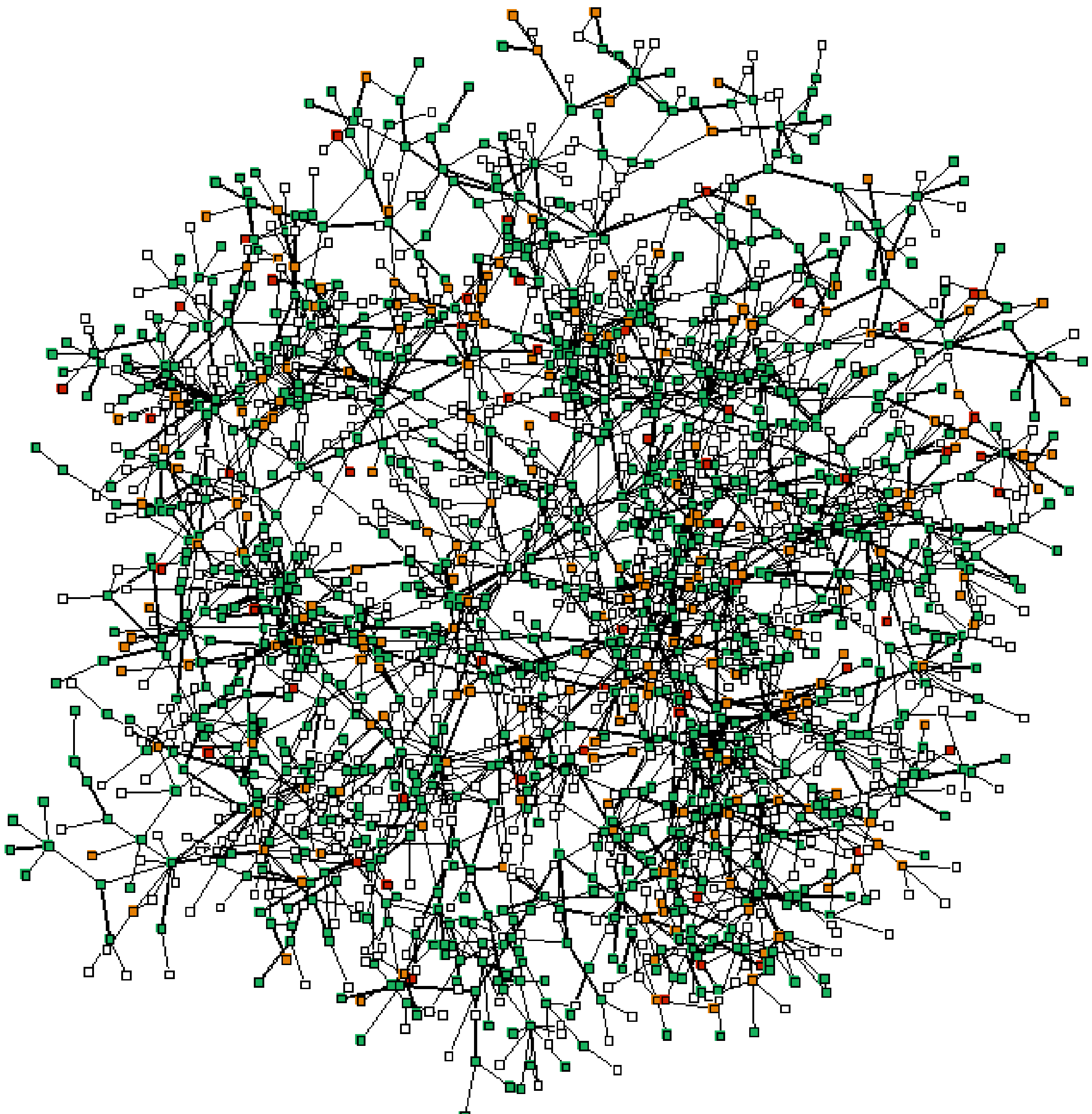
The disordered solution $a = \frac{1}{2}$ is always a stationary solution but it loses its stability below $q_c = \frac{3}{5}$. In that case, the system asymptotically reaches the ordered solution

$$a_{\pm} = \frac{1}{2} \pm \sqrt{\frac{3-5q}{12(1-q)}}$$



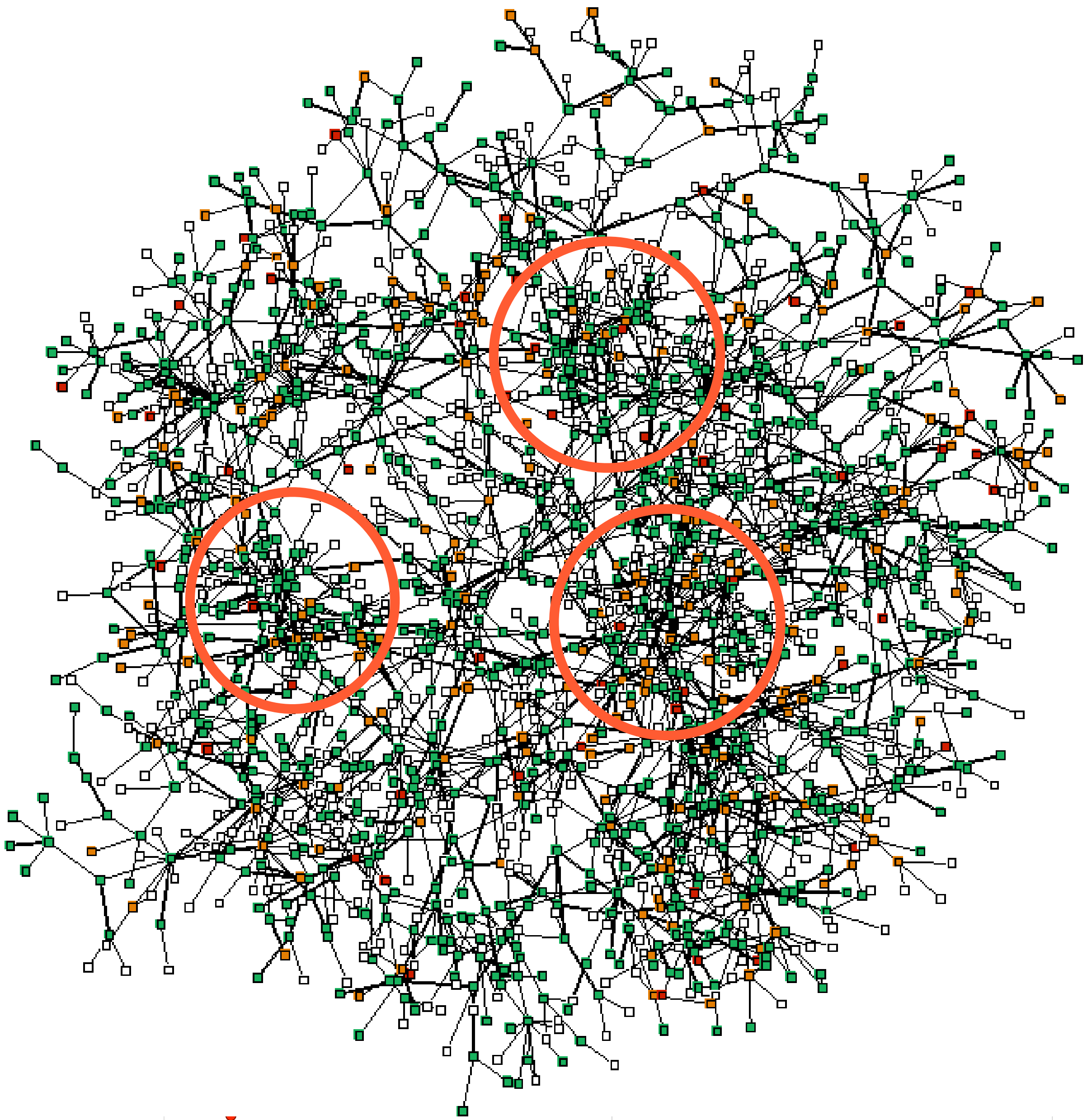
Under the critical value, a collective opinion has emerged due to the *imitation* between neighbouring nodes.

Role of the underlying topology



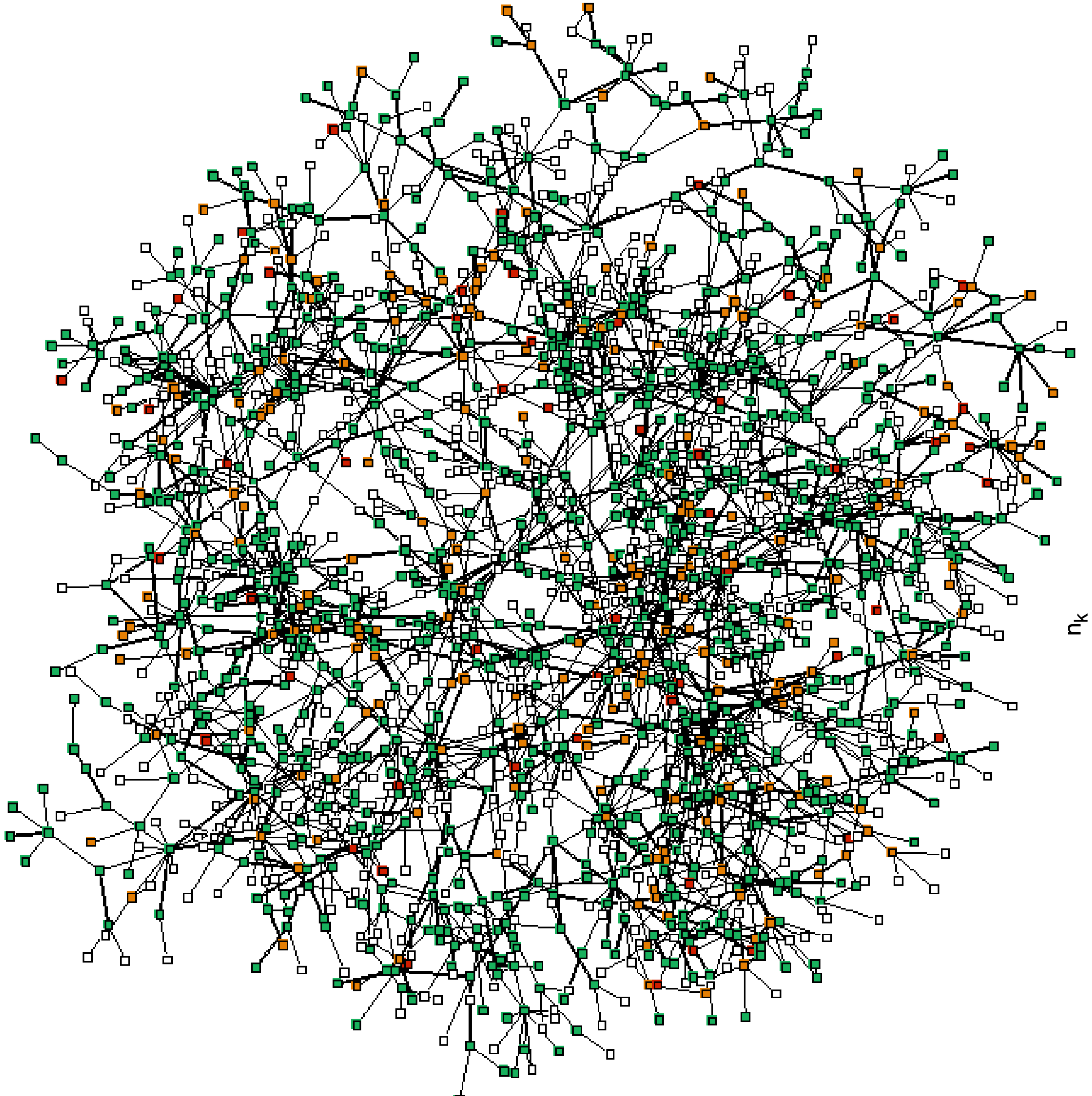
2000 users from a Belgian
Mobile Phone Network
1 day of communication
~ 1 million active users

Role of the underlying topology: modular structures

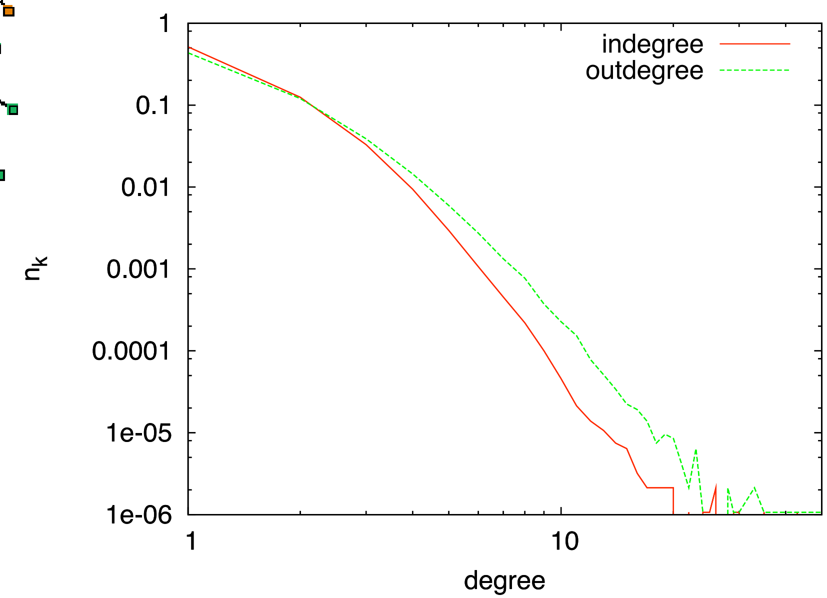


Highly connected communities, while nodes in different communities are sparsely connected

Role of the underlying topology: degree heterogeneity



Broad degree distribution: presence of hubs



Coupled Random Networks

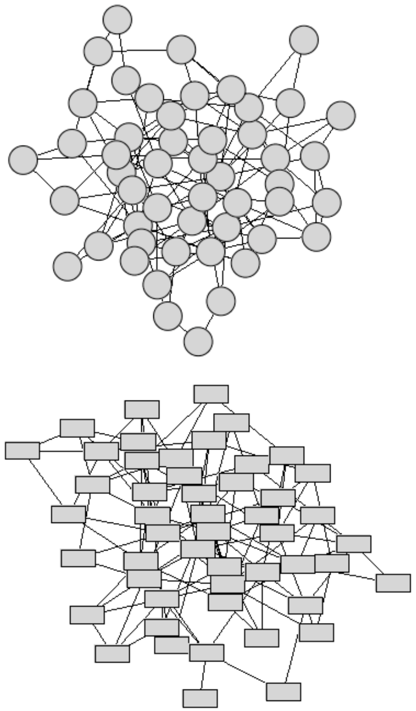
Distinct communities within networks are defined as subsets of nodes which are more densely linked when compared to the rest of the network.

The network is composed of N nodes divided into two types of nodes, 1 and 2. Different types of nodes have a probability p_{cross} to be linked, while nodes of the same type have a probability p_{in} to be linked.

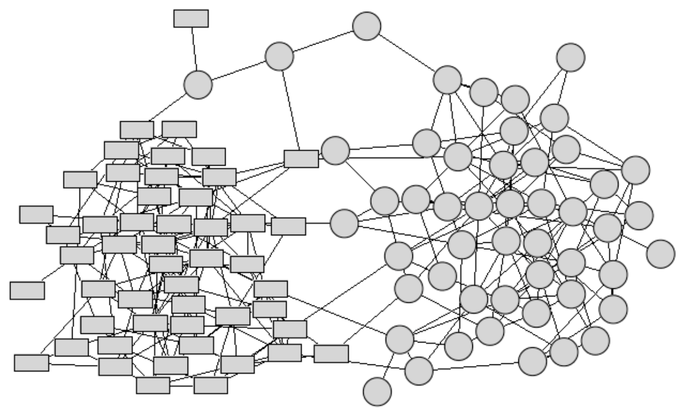
The inter-connectivity between the communities is tunable through the parameter

$$\nu = p_{cross}/p_{in}$$

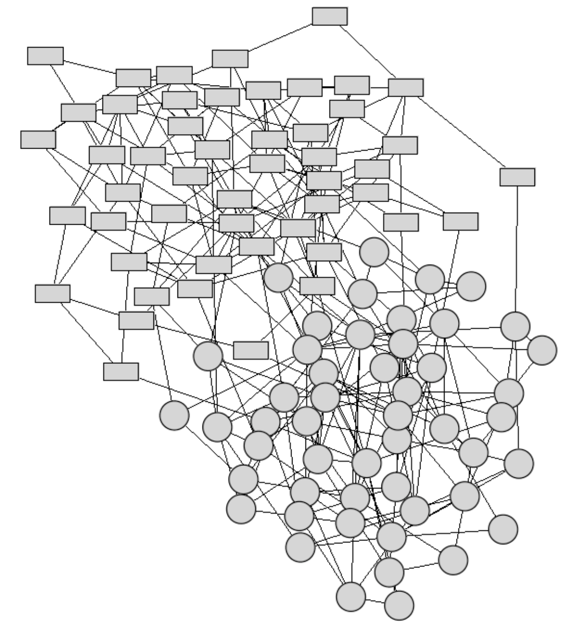
Coupled Random Networks



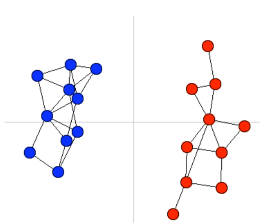
$\nu = 0$

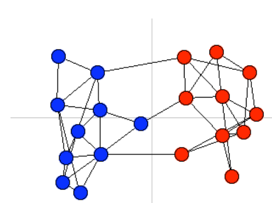


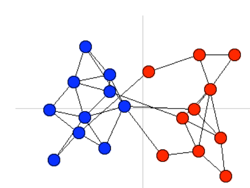
$\nu = 0.05$

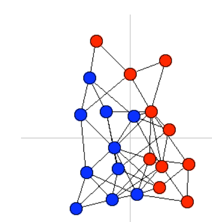


$\nu = 0.1$









The coupled equations of evolution now read

$$
\begin{aligned}
A_{1;t+1} - A_{1;t} = \frac{q}{4} - \frac{qa_1}{2} + (1-q)[\frac{3}{2}\frac{1}{(1+\nu)^2}(a_1^2 b_1 - a_1 b_1^2) \\
+ \frac{\nu^2+2\nu}{(1+\nu)^2}(a_2 a_1 b_1 - a_1 b_2 b_1) + \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}(a_2^2 b_1 - a_1 b_2^2)] \\
A_{2;t+1} - A_{2;t} = \frac{q}{4} - \frac{qa_2}{2} + (1-q)[\frac{3}{2}\frac{1}{(1+\nu)^2}(a_2^2 b_2 - a_2 b_2^2) \\
+ \frac{\nu^2+2\nu}{(1+\nu)^2}(a_1 a_2 b_2 - a_2 b_1 b_2) + \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}(a_1^2 b_2 - a_2 b_1^2)]
\end{aligned}
$$

$$b_i = 1 - a_i$$

And one shows that $a = \frac{1}{2}$ and $a_{\pm} = \frac{1}{2} \pm \sqrt{\frac{3-5q}{12(1-q)}}$
are again solutions of the dynamics

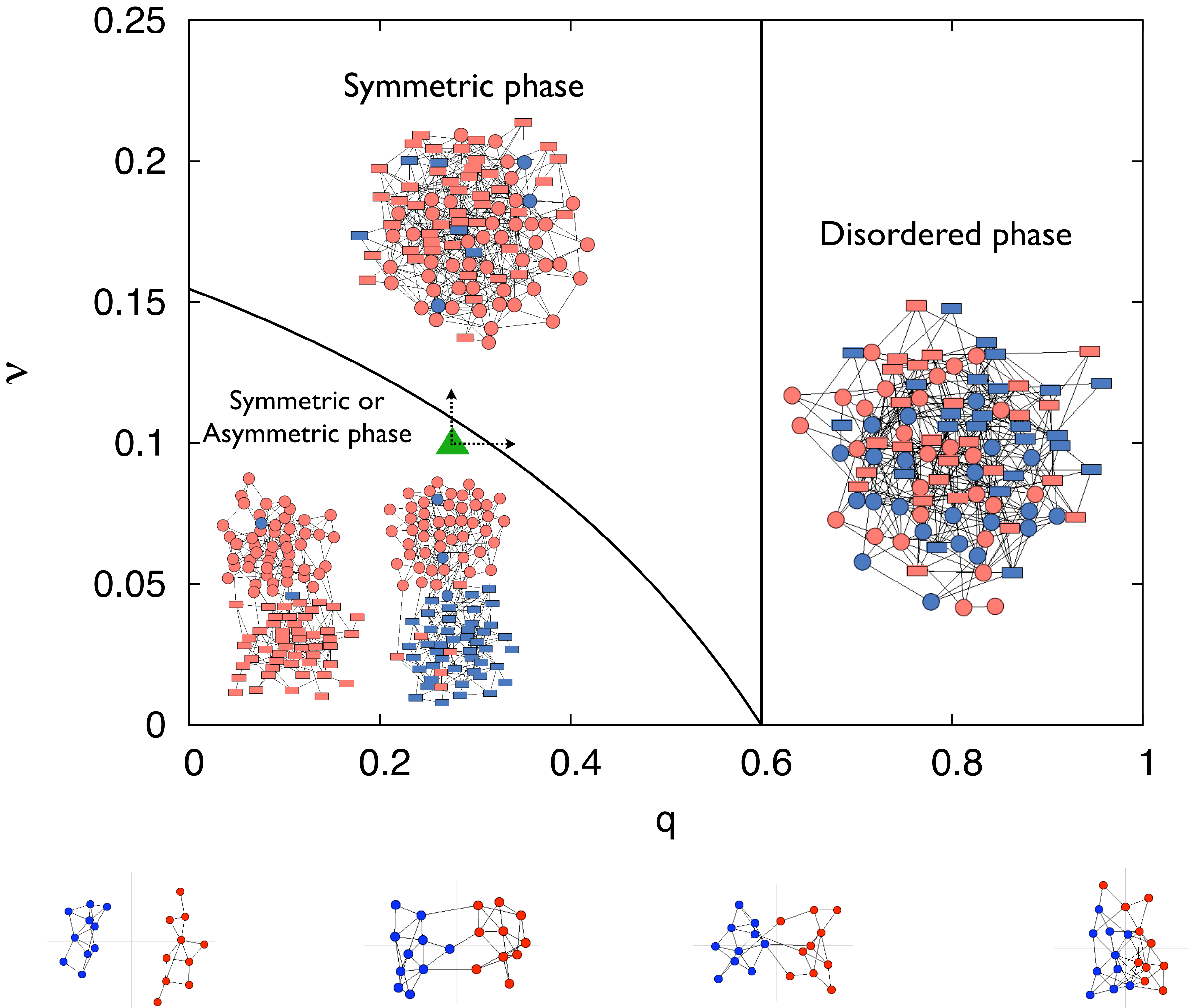
$$q < 3/5$$

But an asymmetric solution may also take place for small enough inter-connectivity. This solution has the form $a_1 = 1/2 + \delta_A$ and $a_2 = 1/2 - \delta_A$, i.e. the two communities reach a different average opinion and one observes the coexistence of two opposite opinions in the system.

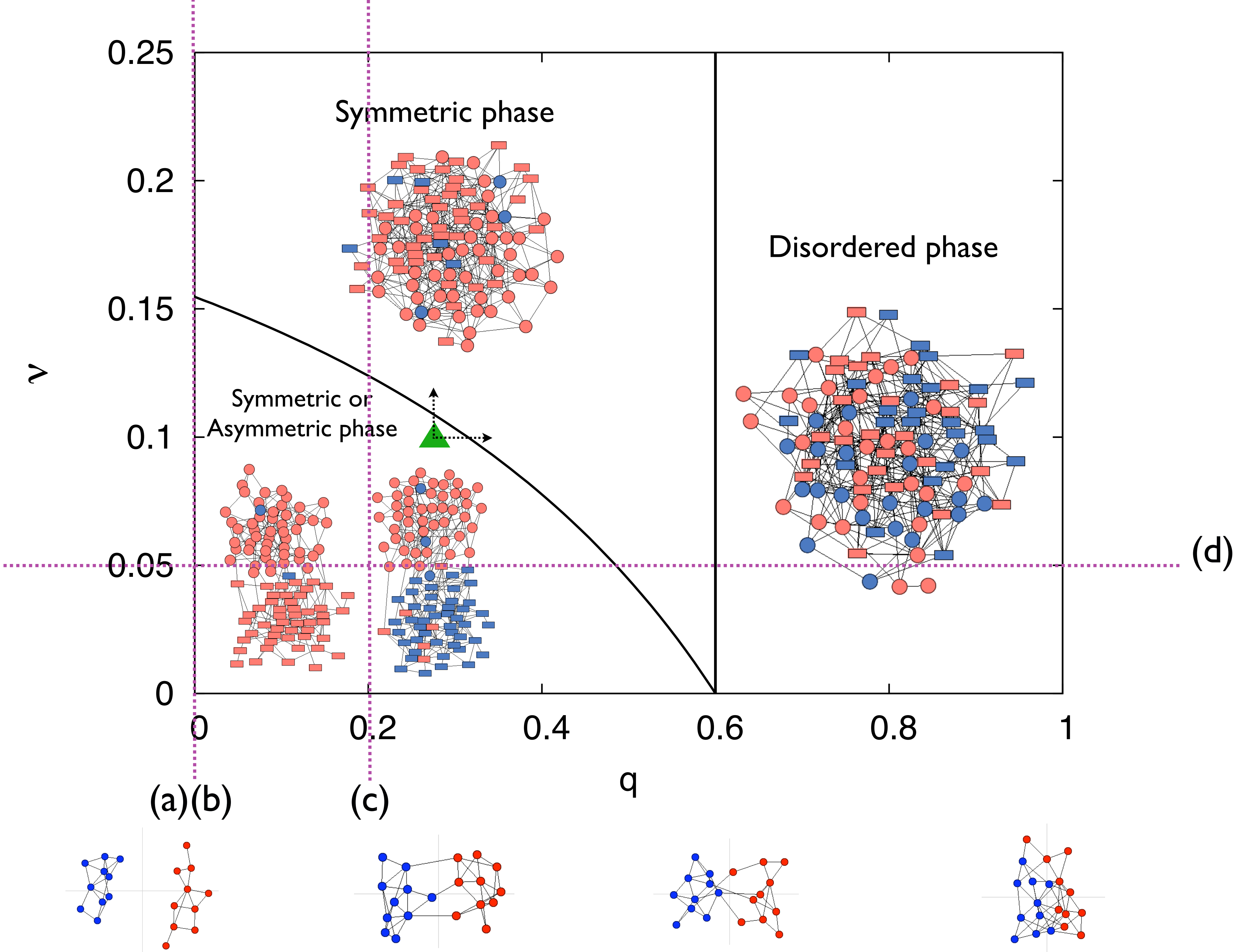
$$\delta_A^2 = \frac{3 - 2\frac{q(1+\nu)^2}{1-q} - 5(\nu^2 + 2\nu)}{12 - 4(\nu^2 + 2\nu)} \rightarrow |a_1 - a_2| > 0$$

This solution ceases to be stable $\nu_c(q) = -1 + 2\sqrt{\frac{3q-3}{7q-9}}$,

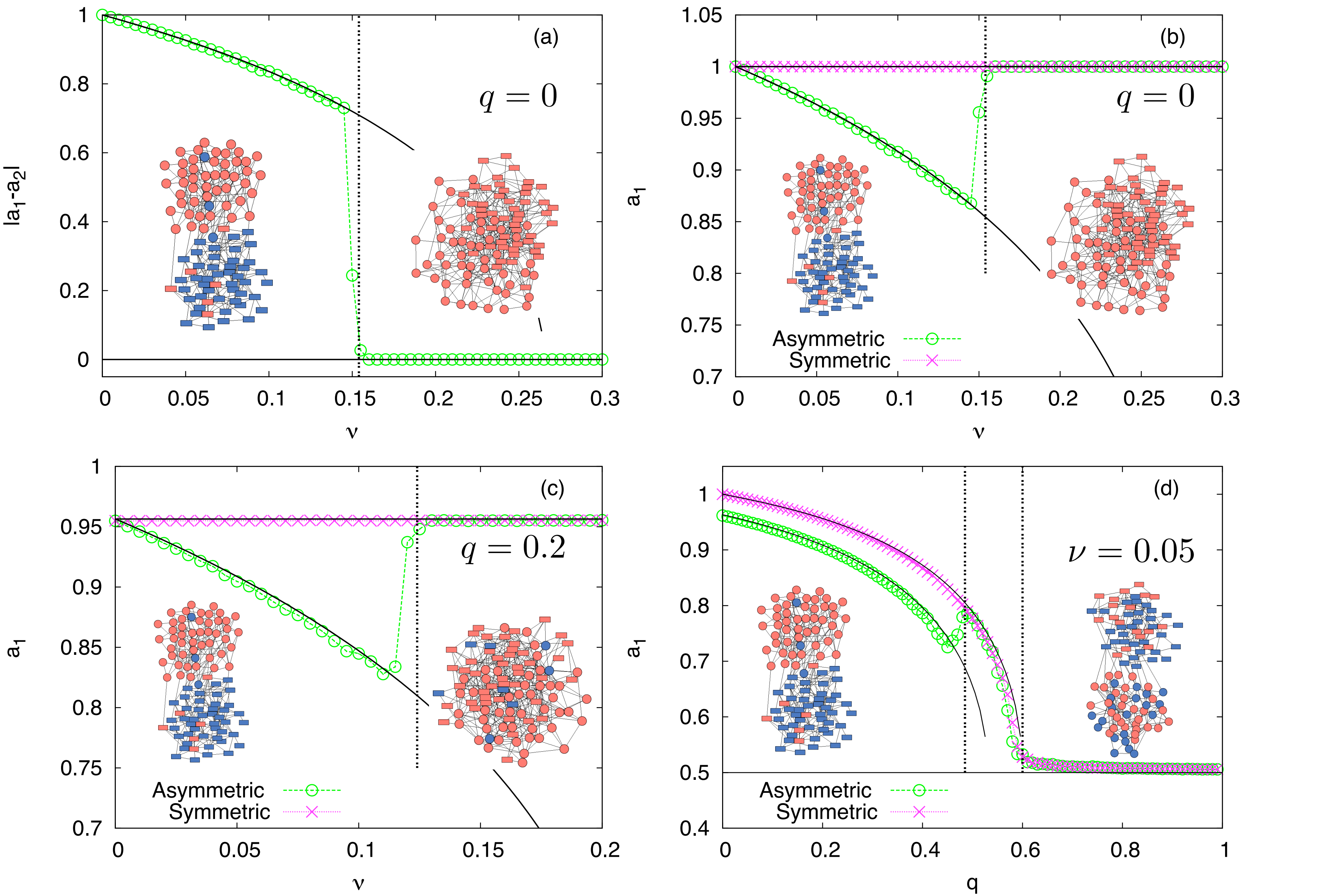
and the transition to the ordered solution is discontinuous at that point.



Symmetric phase
Disordered phase
Symmetric or
Asymmetric phase
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
ν
q



Symmetric phase
Disordered phase
Symmetric or
Asymmetric phase
ν
q
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
(a)(b)
(c)
(d)


(a)
$q = 0$
$|a_1 - a_2|$
ν
(b)
$q = 0$
a_1
ν
Asymmetric
Symmetric
(c)
$q = 0.2$
a_1
ν
Asymmetric
Symmetric
(d)
$\nu = 0.05$
a_1
q
Asymmetric
Symmetric

Some comments

Use of Ising-like models in order to unravel communities in complex networks: minimum resolution

J. Reichardt and S. Bornholdt, *Phys. Rev. Lett.* **93**, 218701 (2004)

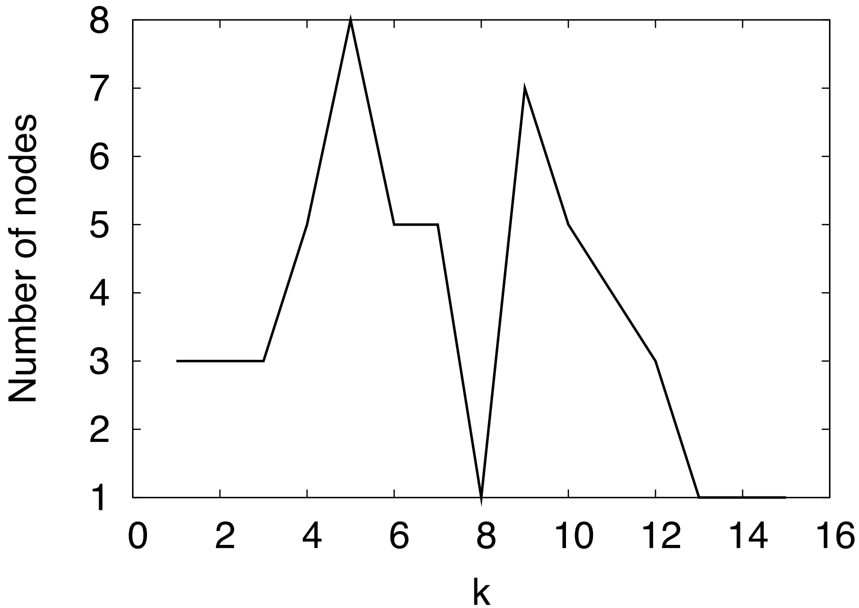
The discontinuity of the transition from the asymmetric state to a symmetric "globalized" state might have radical consequences for systems close to the transition line: the addition of a few links between the communities or a small increase of the fluctuations inside the system may be sufficient in order to drive the system out of the asymmetric state: niche market, language dynamics, etc.

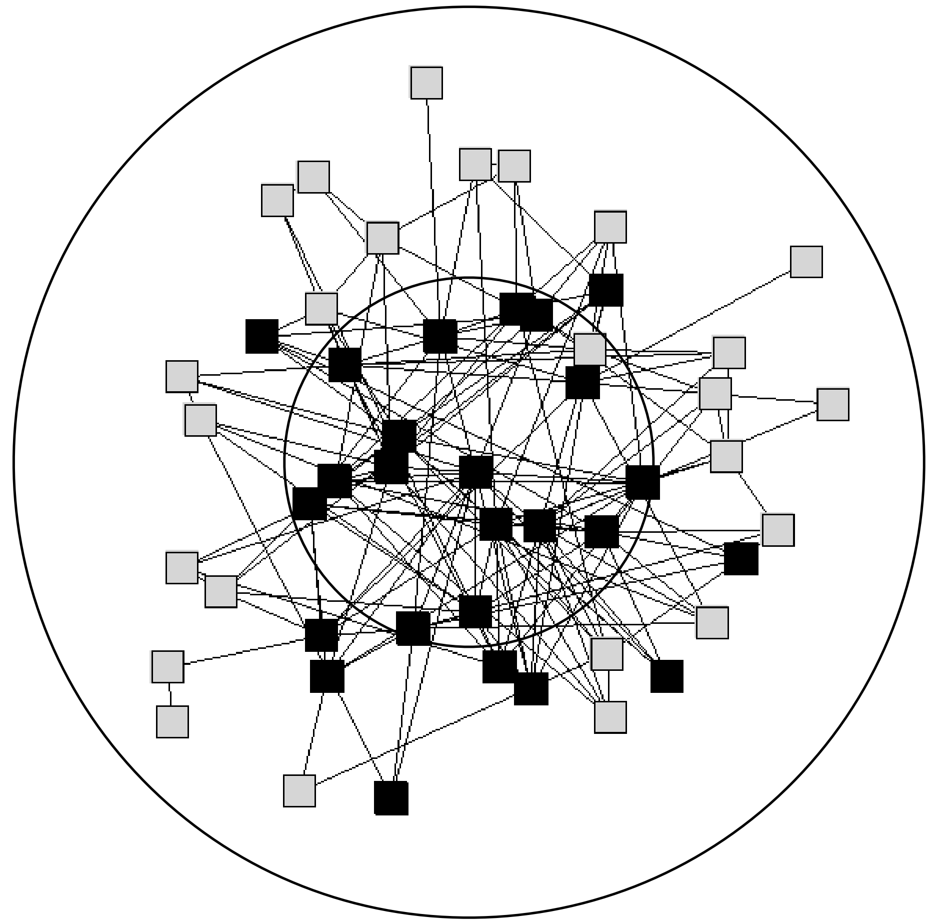
Role of the degree heterogeneity

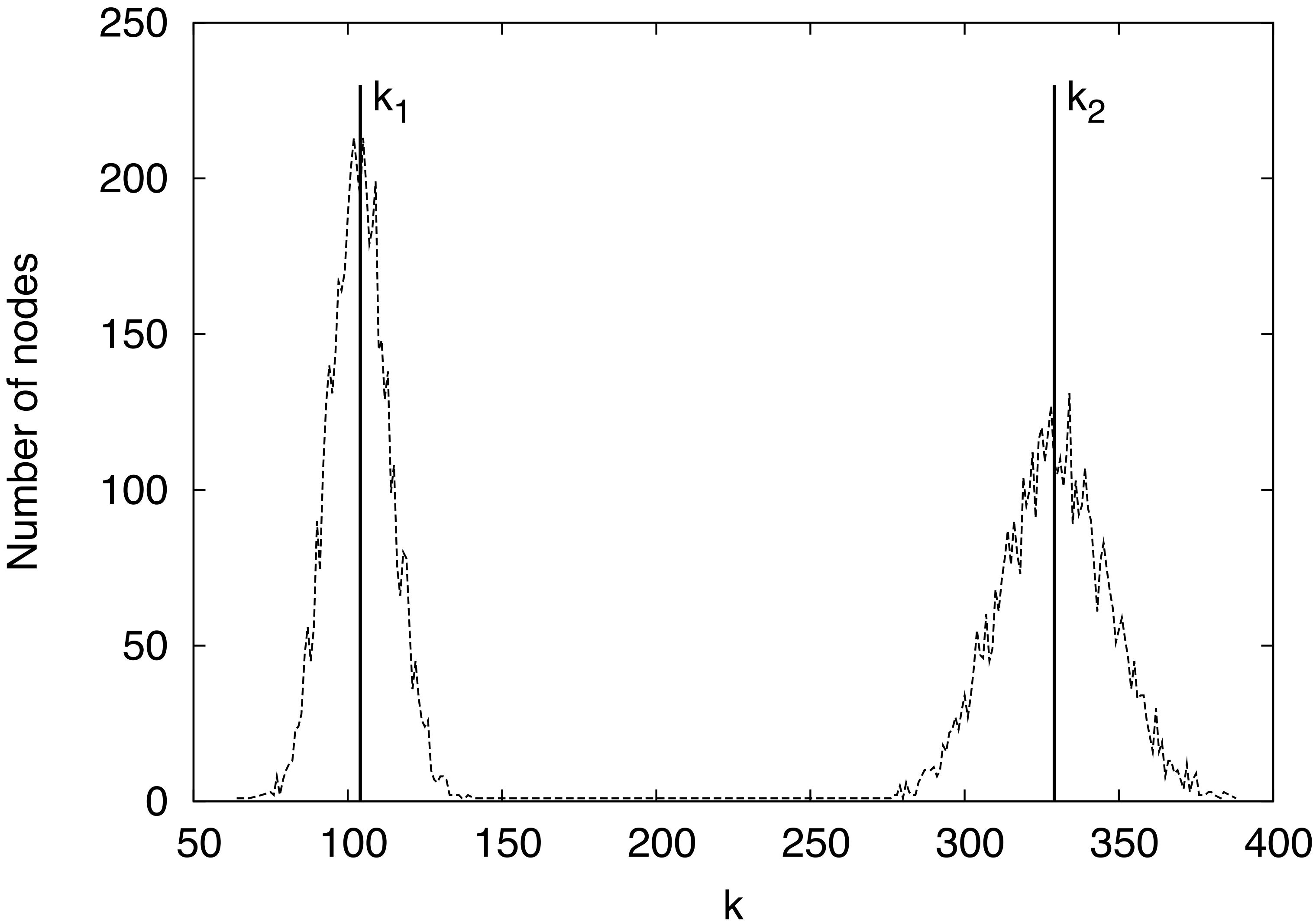
Dichotomous networks: there are two kinds of nodes, each kind *i* being characterised by a degree k_i

Hidden variable $p_1 = 0.05$

Hidden variable $p_2 = 0.25$





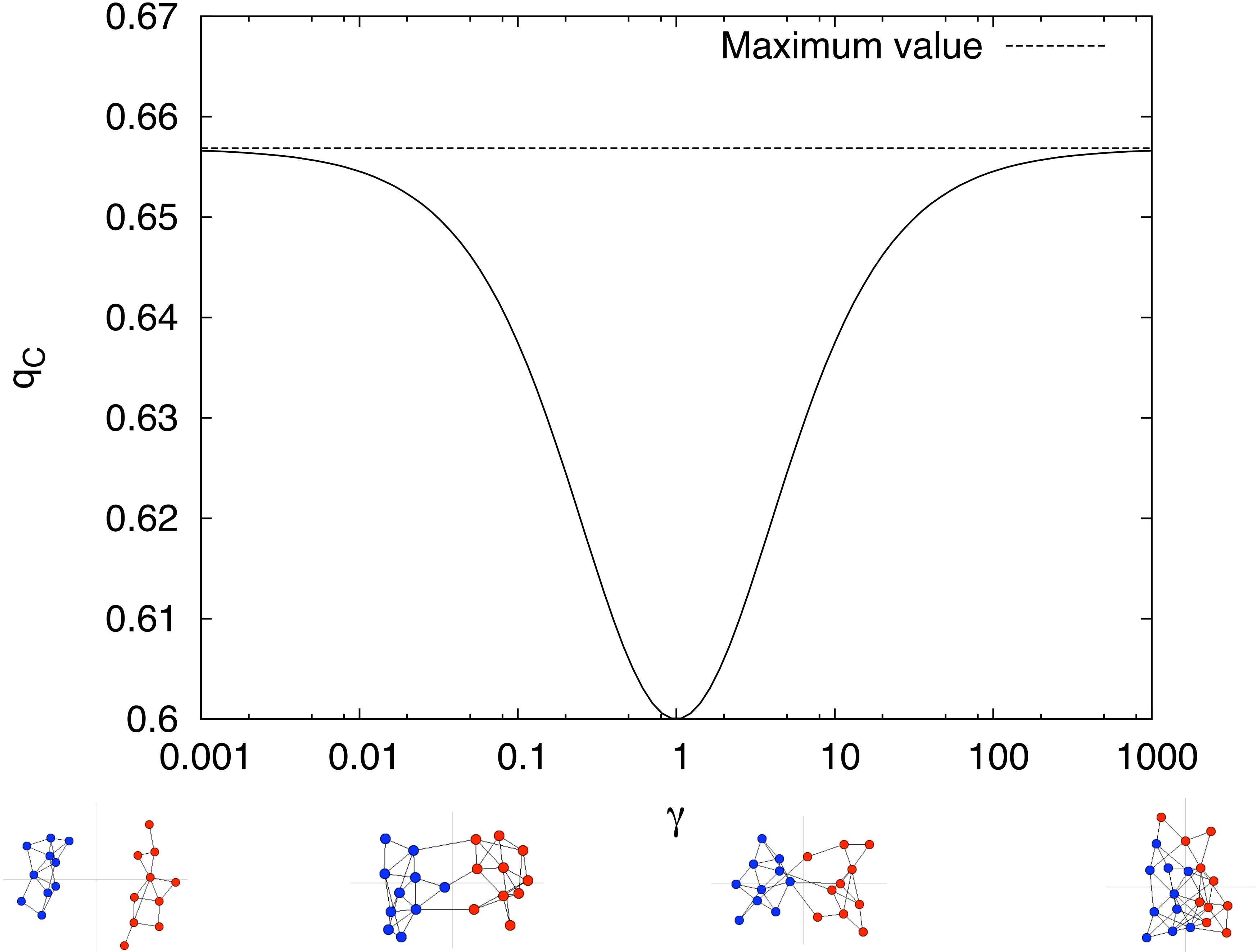


Degree heterogeneity: $\gamma = k_2/k_1$

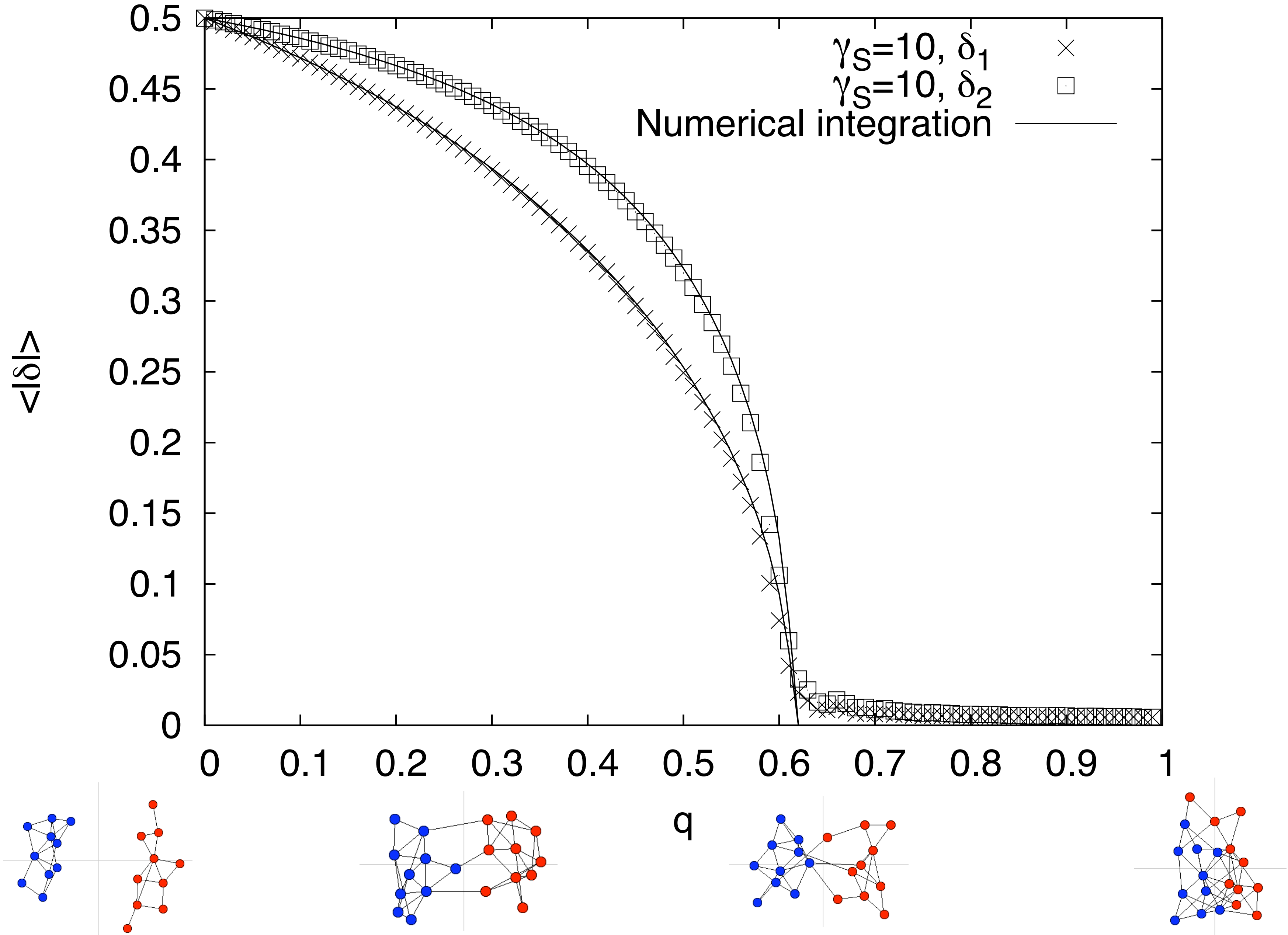
The coupled equations of evolution now read

$$
\begin{aligned}
A_{1;t+1} - A_{1;t} &= \frac{q}{4} - \frac{qa_1}{2} + \frac{(1-q)}{2(1+\gamma)^2}[3(a_1^2 b_1 - a_1 b_1^2) \\
&+ 2(1+2\gamma)(a_2 a_1 b_1 - a_1 b_2 b_1) \\
&+ (2\gamma + \gamma^2)(a_2^2 b_1 - a_1 b_2^2)] \\
A_{2;t+1} - A_{2;t} &= \frac{q}{4} - \frac{qa_2}{2} + \frac{(1-q)}{2(1+\gamma)^2}[3\gamma^2(a_2^2 b_2 - a_2 b_2^2) \\
&+ 2(2\gamma + \gamma^2)(a_1 a_2 b_2 - a_2 b_1 b_2) \\
&+ (1+2\gamma)(a_1^2 b_2 - a_2 b_1^2)], \qquad (11)
\end{aligned}
$$

Degree heterogeneity displaces the location of the transition



Degree heterogeneity implies a *non-equipartition* of the opinion between species



Some comments

The location of the transition depends on the degree heterogeneity

→ A change of the underlying topology (degree distribution) might trigger a phase transition

Opinion heterogeneity

→ Interplay between topology and “magnetisation”
In real data?