Dynamics and Multiscale Modular Structure in Networks

R. Lambiotte
Institute for Mathematical Sciences
Imperial College London

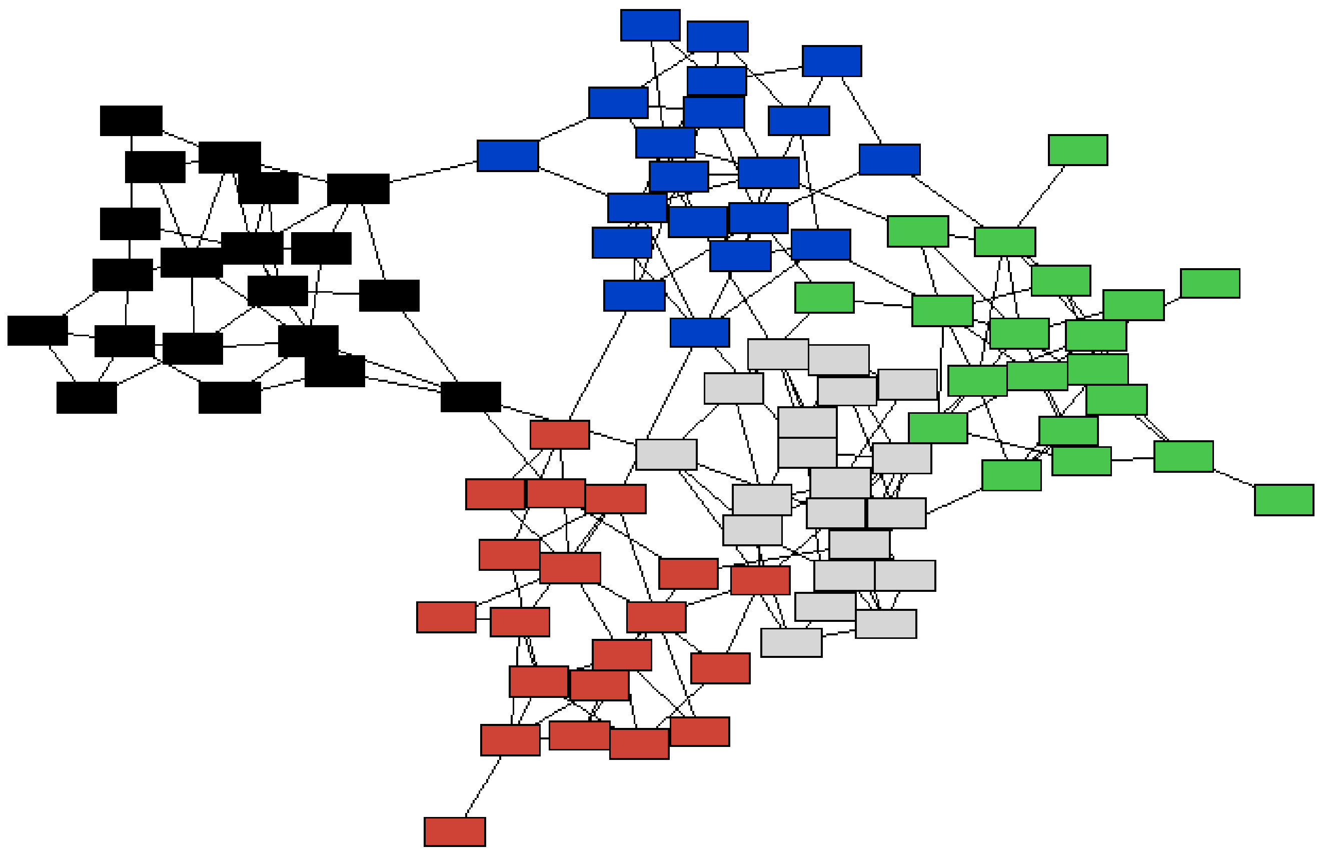
with M. Barahona and J.-C. Delvenne

Imperial College London
100 years of living science
100

R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*

Modular Networks

Most networks are very inhomogeneous and are made of modules:
many links within modules and a few links between different modules



Internet

Power grids

Food webs

Metabolic networks

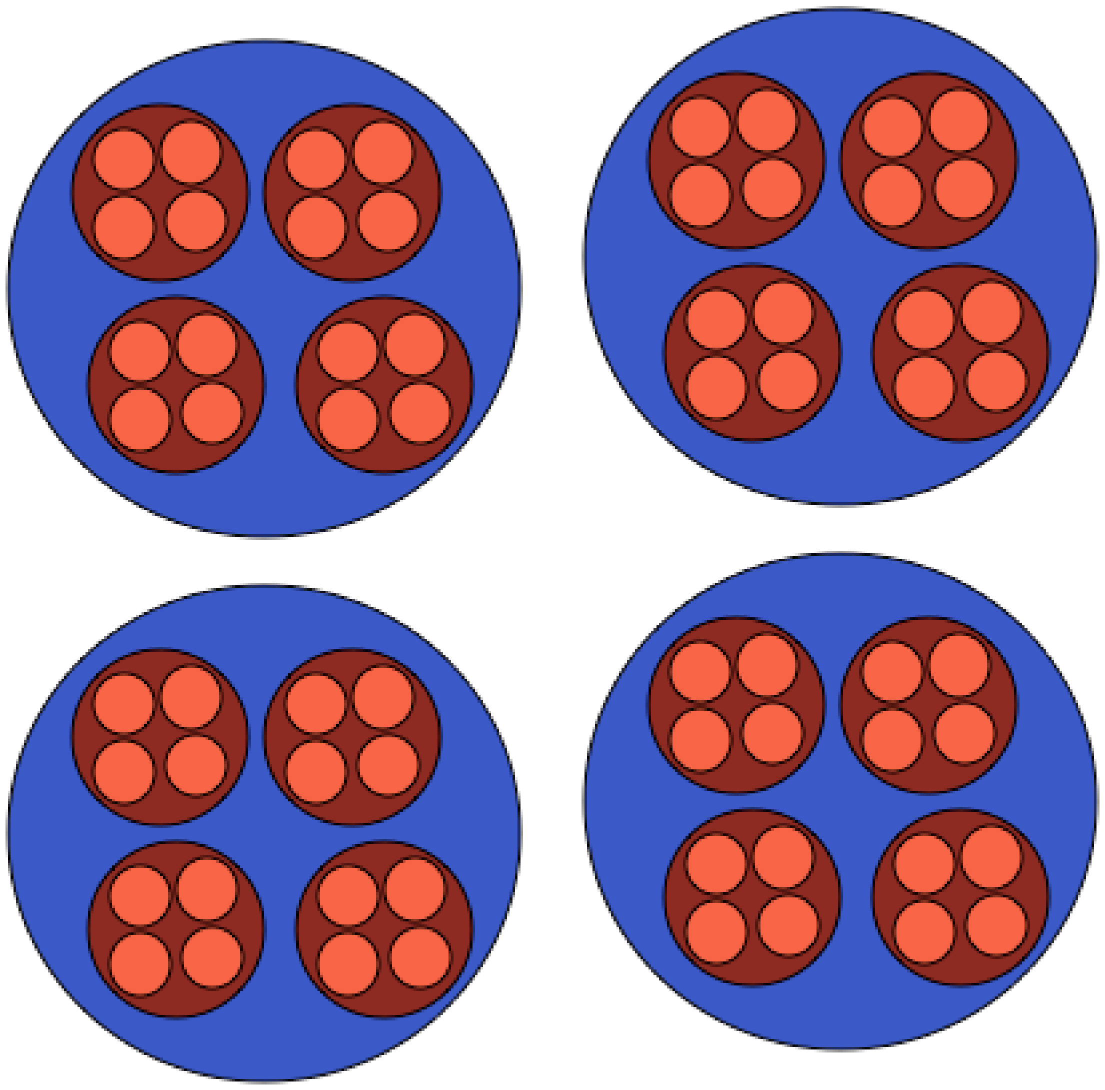
Social networks

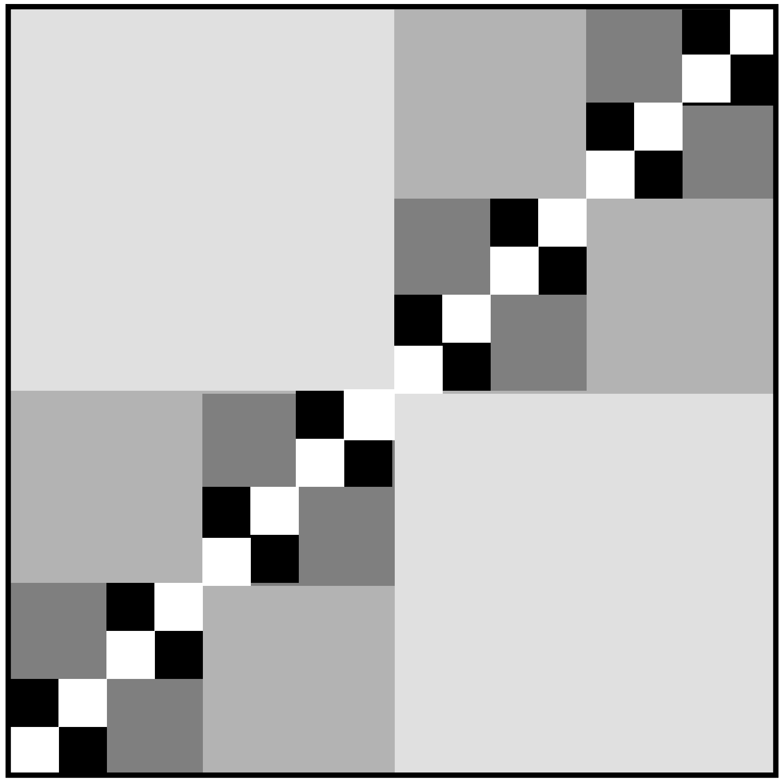
The brain

Etc.

Modular Networks

Networks have a hierarchical structure: modules within modules

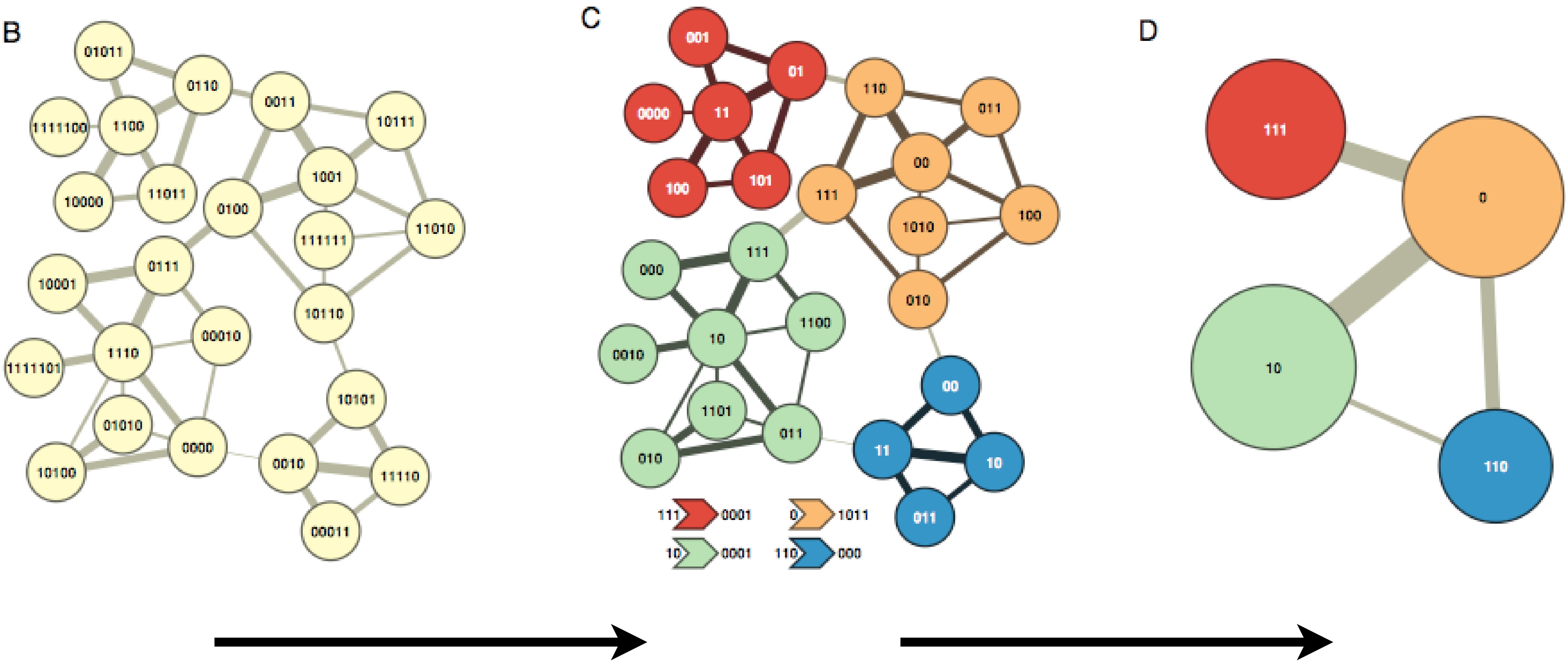




Simon, H. (1962). The architecture of complexity. Proceedings of the American Philosophical Society, 106, 467–482.

Modular Networks

Uncovering communities/modules helps to understand the structure of the network, to uncover similar nodes and to draw a readable map of the network (when N is large).



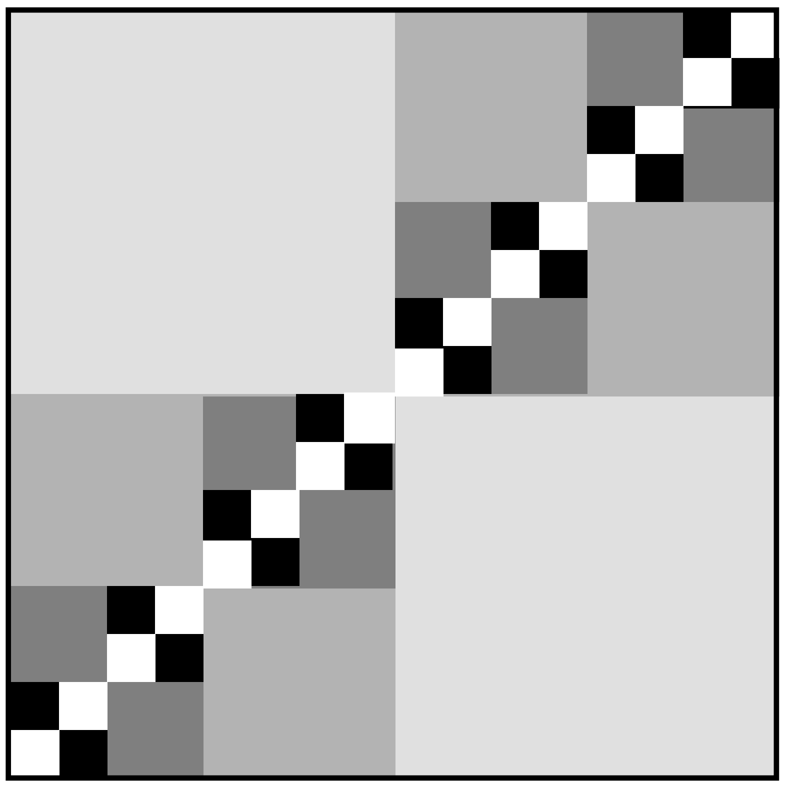
Find a partition of the network into communities

Coarse-grained description

Martin Rosvall and Carl T. Bergstrom, PNAS 105, 1118 –1123 (2008)

Modular Networks and dynamics

Many networks are "modular" and have a hierarchical structure: modules within modules



How does such modularity affect dynamics?

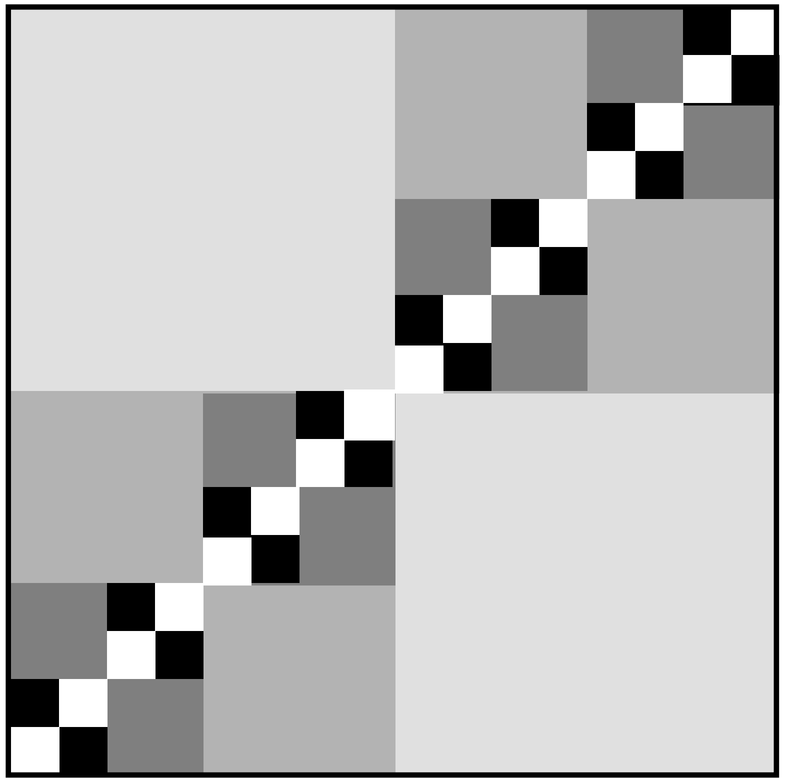
A. Arenas, A. Diaz-Guilera and C.J. Pérez-Vicente, *Phys. Rev. Lett.* **96**, 114102 (2006).
R. Lambiotte, M. Ausloos and J.A. Holyst, *Phys. Rev. E* **75**, 030101(R) (2007).

Is it possible to uncover those modules in large networks?

NG, GN, Walktrap, clique-percolation, Simulated Annealing, etc.

Modular Networks and dynamics

Many networks are “modular” and have a hierarchical structure:
modules within modules



How does such modularity affect dynamics?

A. Arenas, A. Diaz-Guilera and C.J. Pérez-Vicente, *Phys. Rev. Lett.* **96**, 114102 (2006).
R. Lambiotte, M. Ausloos and J.A. Holyst, *Phys. Rev. E* **75**, 030101(R) (2007).

Is it possible to use dynamics to characterize (and uncover?) the modular structure of a network?

e.g. Walktrap (RW exploration)

Is it possible to uncover those modules in large networks?

NG, GN, Walktrap, clique-percolation, Simulated Annealing, etc.

Notations

Let us focus on an unweighted, undirected network

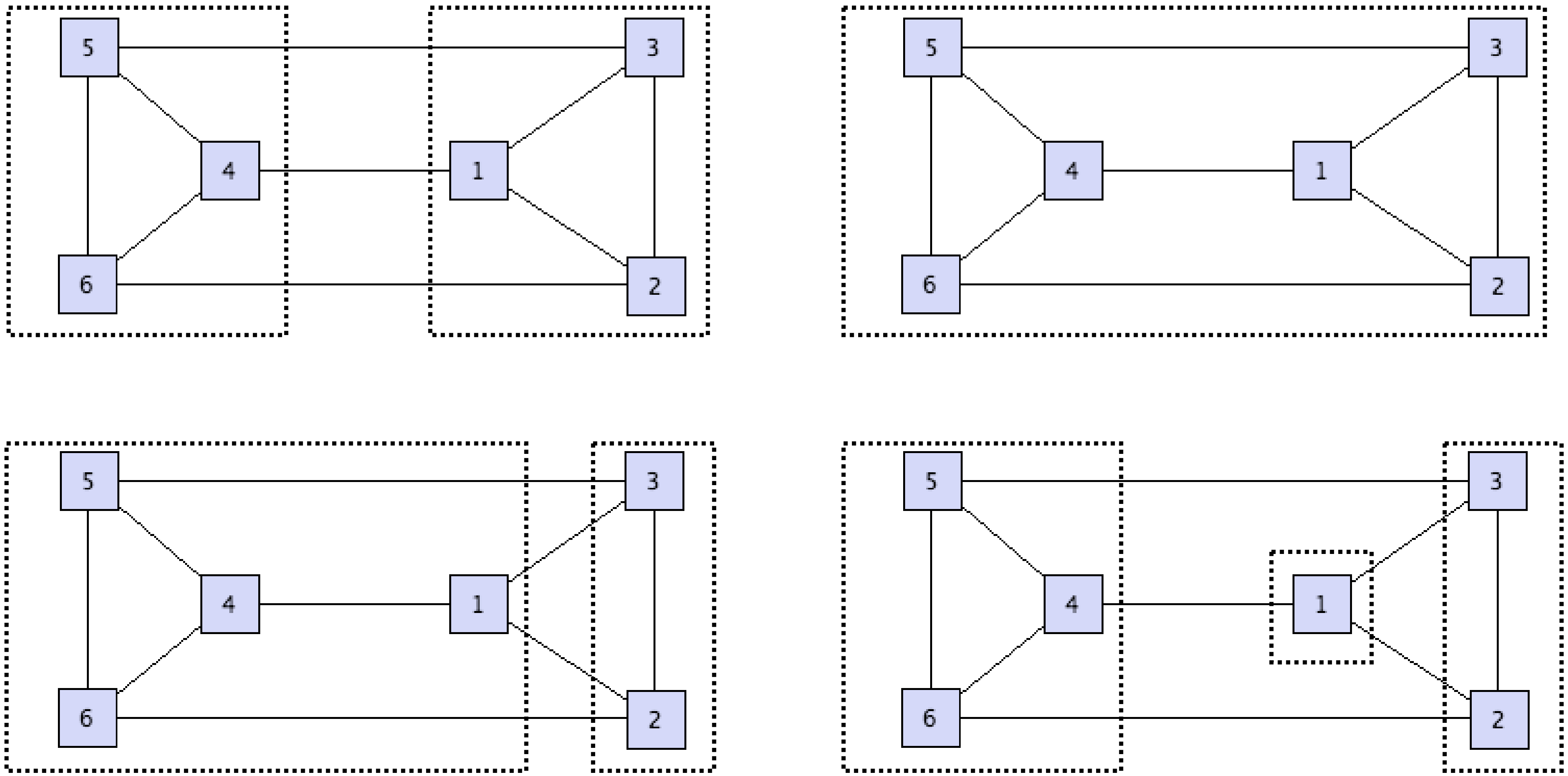
A_{ij} adjacency matrix

$k_i = \sum_j A_{ij}$ *degree* of *i*

$m = \frac{1}{2} \sum_i k_i$ total number of links

Quality of a partition

What is the best partition of a network into modules?



....

Modularity

Q = fraction of edges within communities - expected fraction of such edges

Let us attribute each node *i* to a community c_i

$$Q = \frac{1}{2m} \sum_{i,j} \Big[A_{ij} - P_{ij} \Big] \delta(c_i, c_j)$$

$$Q \in [-1, 1]$$

$$P_{ij} = \frac{k_i k_j}{2m}$$

expected number of links between i and j

$$\rightarrow \quad Q_C = \frac{1}{2m} \sum_{i,j} \Big[A_{ij} - k_i k_j / 2m \Big] \delta(c_i, c_j)$$

Modularity

Optimising modularity uncovers one partition

What about sub (or hyper)-communities in a hierarchical network?

Reichardt & Bornholdt

$$Q_\gamma = \frac{1}{2m} \sum_{i,j} \left[A_{ij} - \gamma P_{ij} \right] \delta(c_i, c_j)$$

Arenas et al.

$$Q(A_{ij} + r I_{ij})$$

Tuning parameters allow to uncover communities of different sizes

Reichardt & Bornholdt different of Arenas, except in the case of a regular graph where

$$\gamma = 1 + r / \langle k \rangle$$

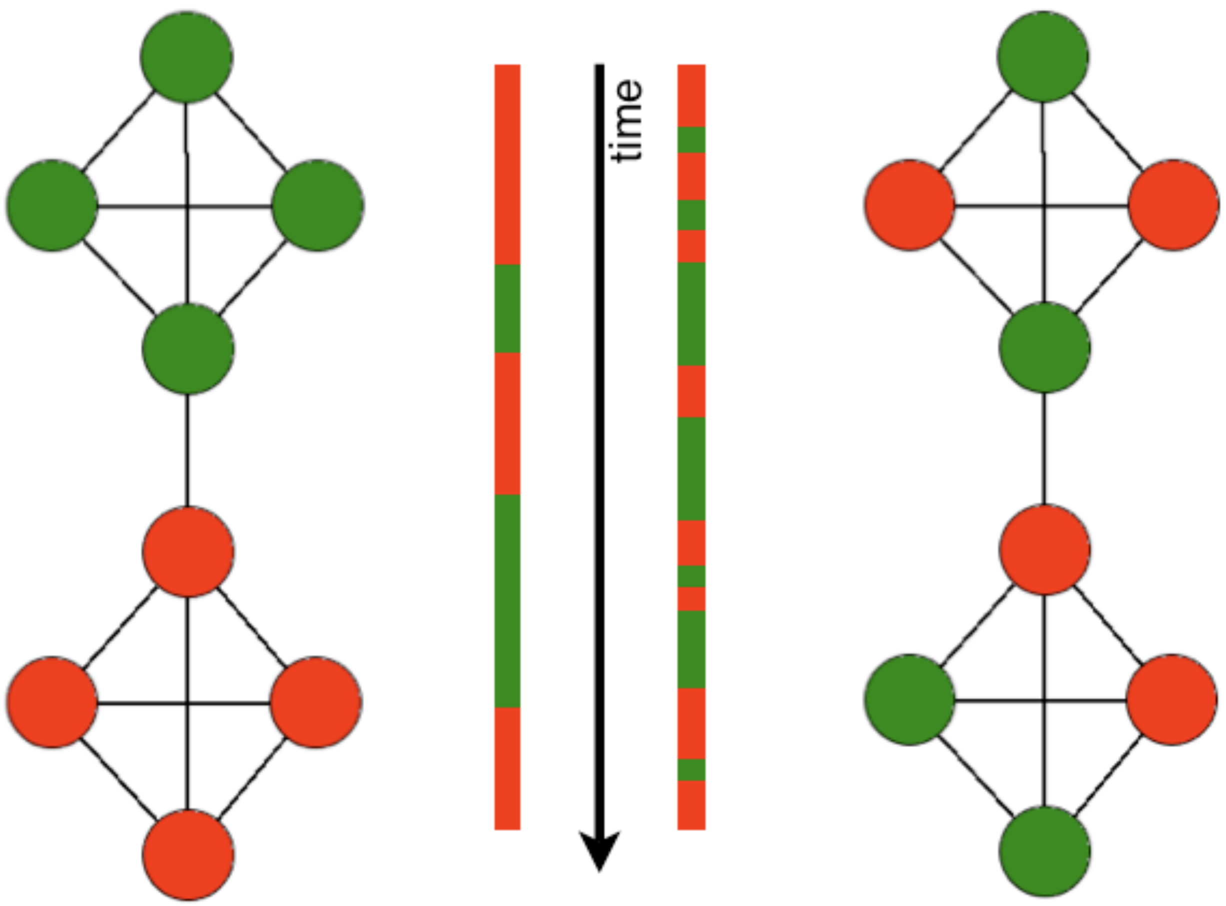
*J. Reichardt and S. Bornholdt, Phys. Rev. E **74**, 016110 (2006). Statistical mechanics of community detection*

*A Arenas, A Fernandez, S Gomez, New J. Phys. **10**, 053039 (2008). Analysis of the structure of complex networks at different resolution levels*

Stability

The quality of a partition is determined by the patterns of a flow within the network: a flow should be trapped for long time periods within a community before escaping it.

The stability of a partition is defined by the statistical properties of a random walker moving on the graph:



J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. arXiv:0812.1811.

Stability

The quality of a partition is determined by the patterns of a flow within the network: a flow should be trapped for long time periods within a community before escaping it.

The stability of a partition is defined by the statistical properties of a random walker moving on the graph:

$$R(t) = \sum_{C \in \mathcal{P}} P(C, t_0, t_0 + t) - P(C, t_0, \infty)$$

$P(C, t_0, t_0 + t)$ probability for a walker to be in the same community at times t_0 and $t_0 + t$ when the system is at equilibrium

$P(C, t_0, \infty)$ probability for two independent walkers to be in C (ergodicity)

J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. arXiv:0812.1811.

Stability

Let us consider a continuous-time random walk with Poisson waiting times

$$\dot{p}_i = \sum_j \frac{A_{ij}}{k_j} p_j - p_i \quad \xrightarrow{\text{equilibrium}} \quad p_i^* = k_i/2m$$

$$R(t) = \sum_{i,j} \left[\left(e^{t(B-I)} \right)_{ij} \frac{k_j}{2m} - \frac{k_i k_j}{(2m)^2} \right] \delta(c_i, c_j)$$

$$B_{ij} = A_{ij}/k_j$$

Probability that a walker is in the same community initially and at time t

Same probability for independent walkers

R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*

Stability: time as a resolution parameter

What are the optimal partitions of R_t?

t=0 $$R(0) = 1 - \sum_{i,j} \frac{k_i k_j}{(2m)^2} \delta(c_i, c_j)$$ → Communities=single nodes

t small $$R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$$

favours single nodes — modularity

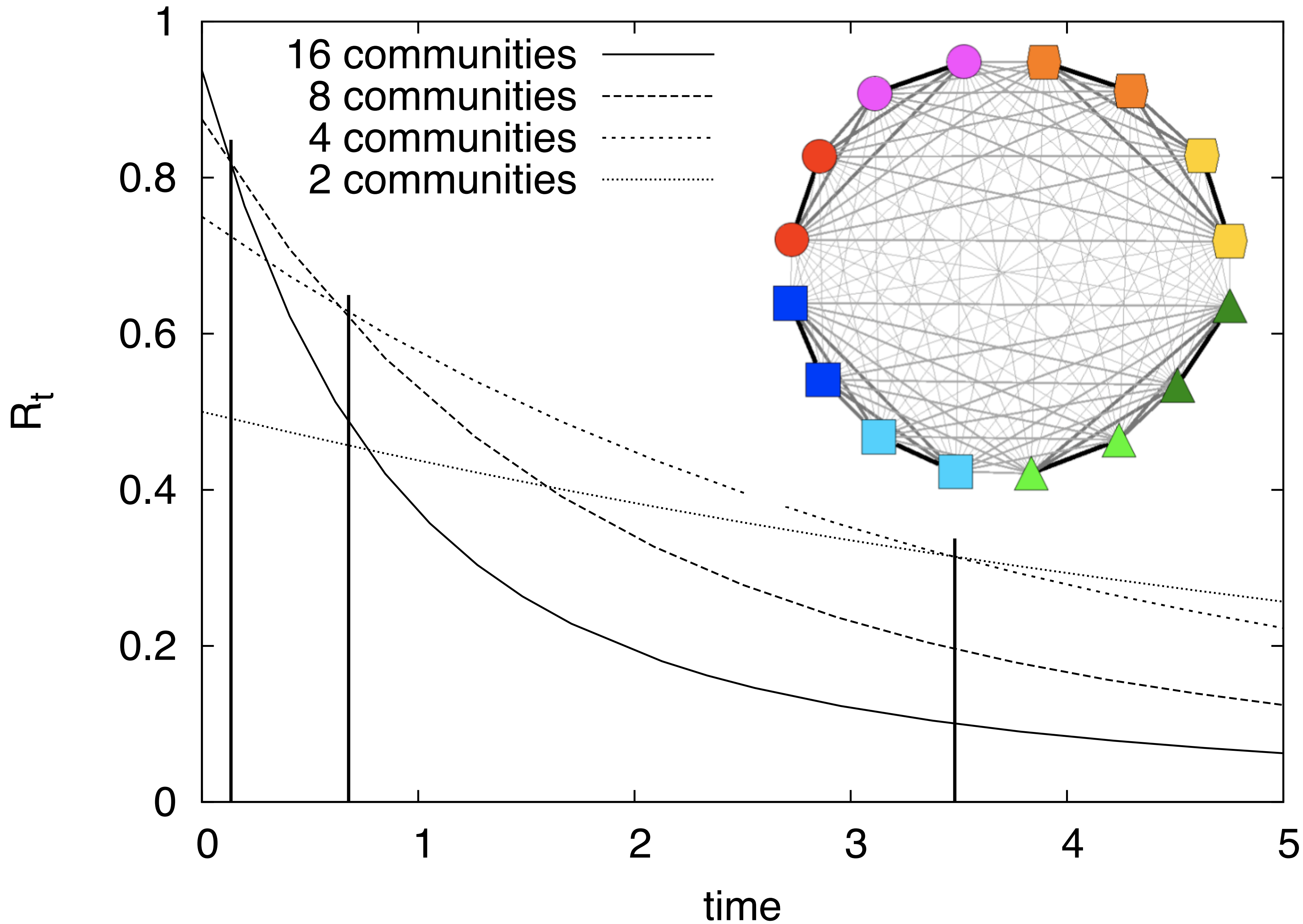
!! Q_t equivalent to the Hamiltonian formulation of Reichardt and Bornholdt (t=1/gamma)

....

When t goes to infinity, the optimal partition is made of 2 communities (by spectral decomposition)

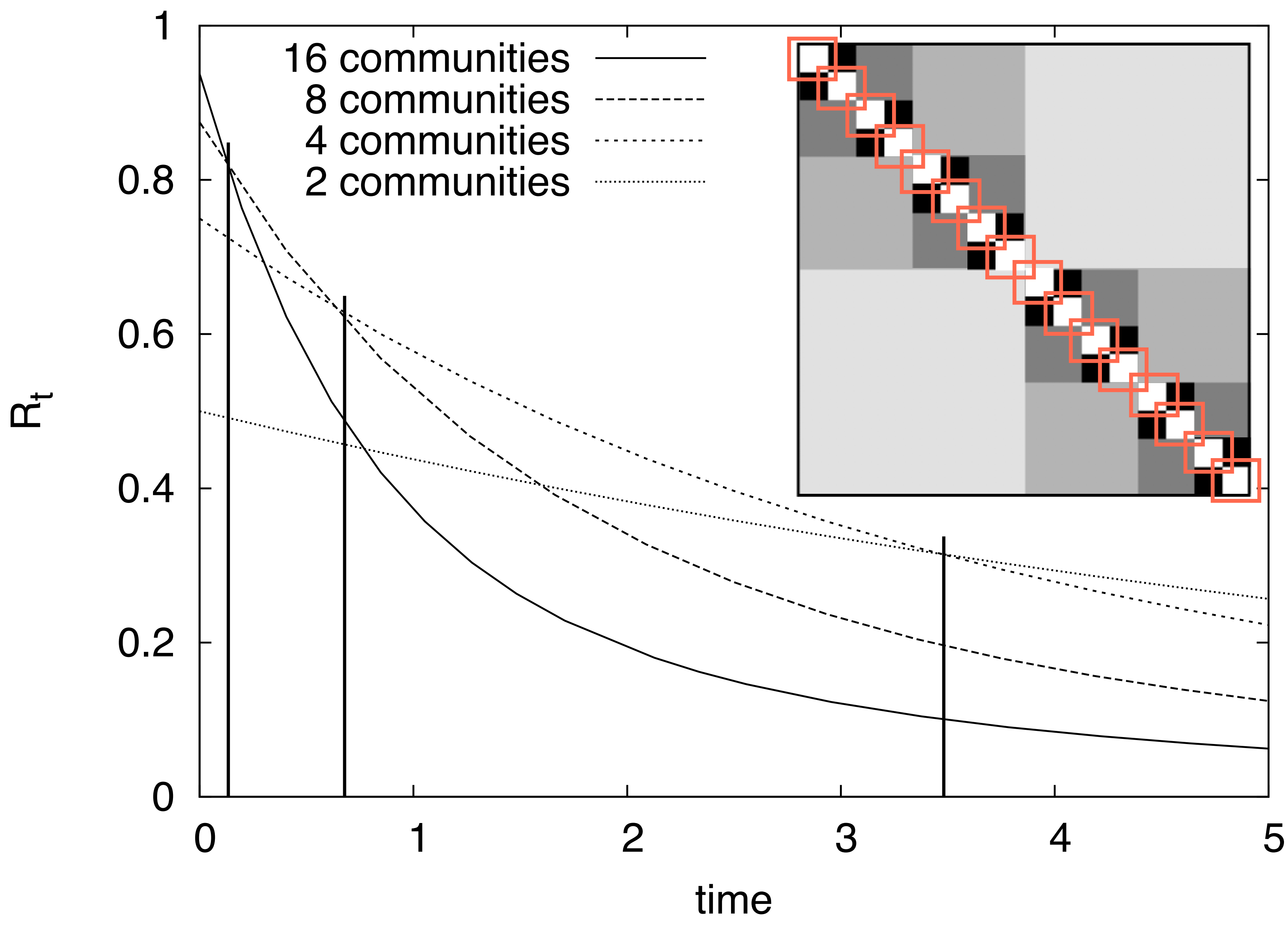
Stability: time as a resolution parameter

Time is a “resolution parameter”: larger and larger communities when time is increased



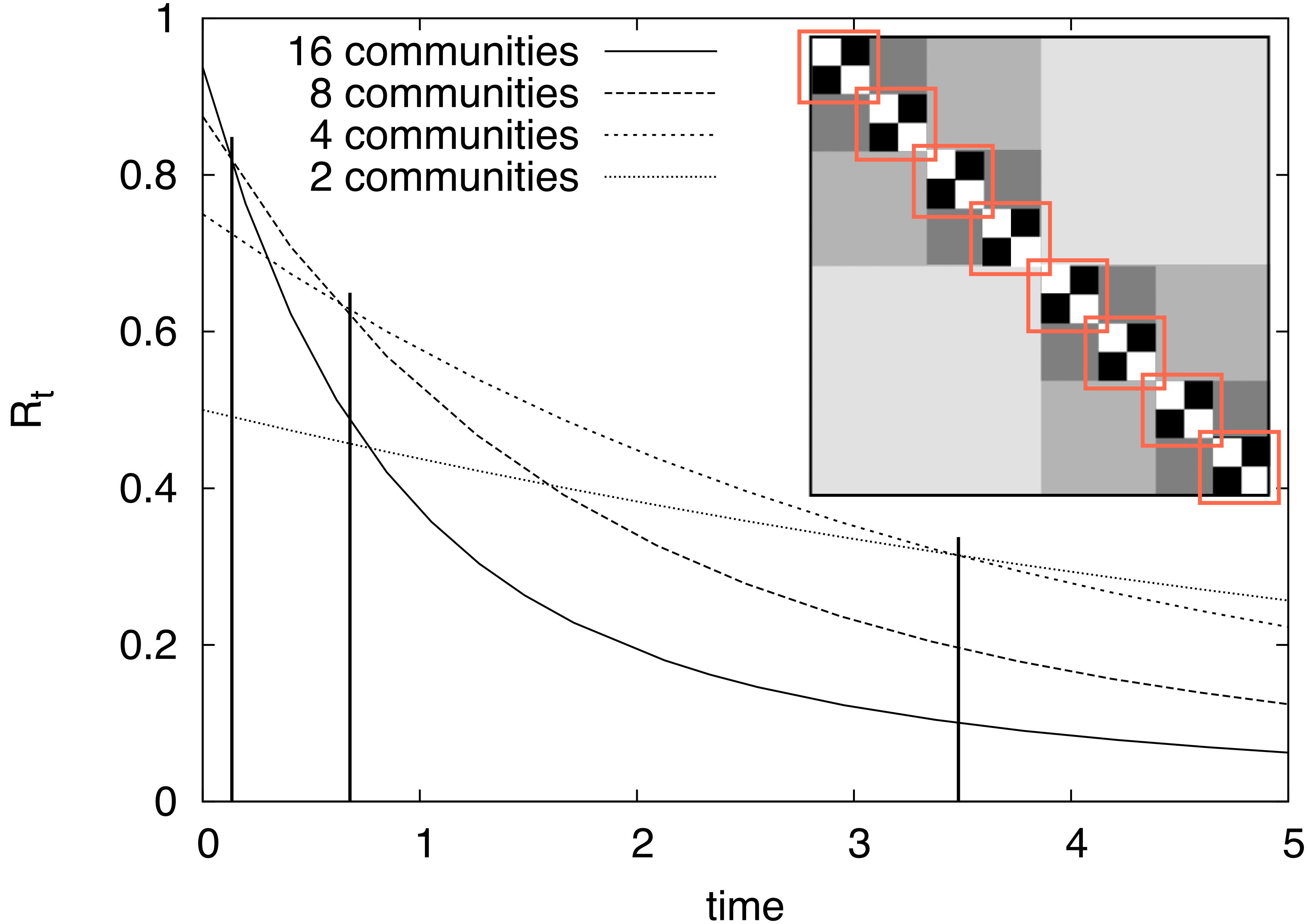
Stability: time as a resolution parameter

Time is a "resolution parameter": larger and larger communities when time is increased



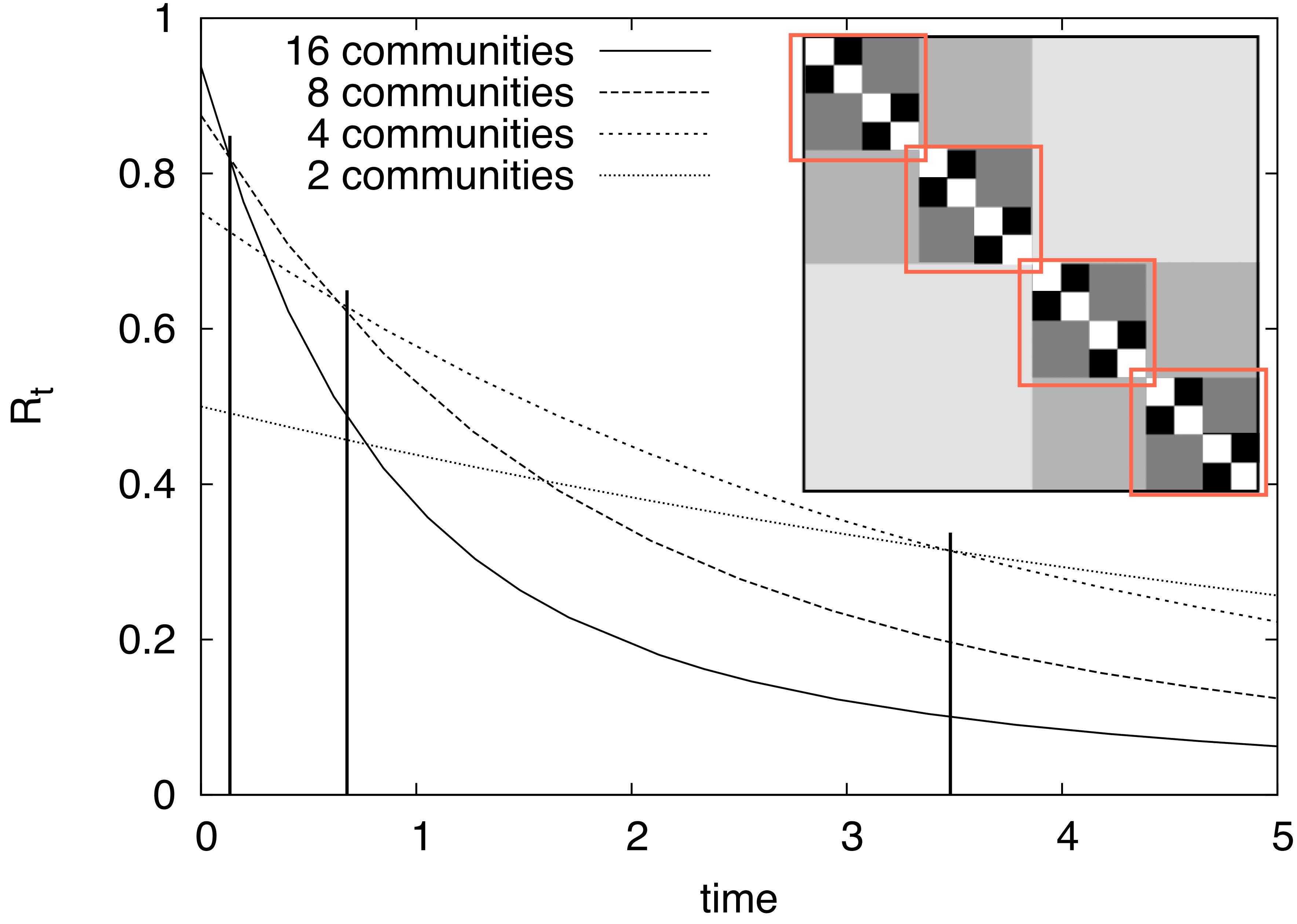
Stability: time as a resolution parameter

Time is a "resolution parameter": larger and larger communities when time is increased



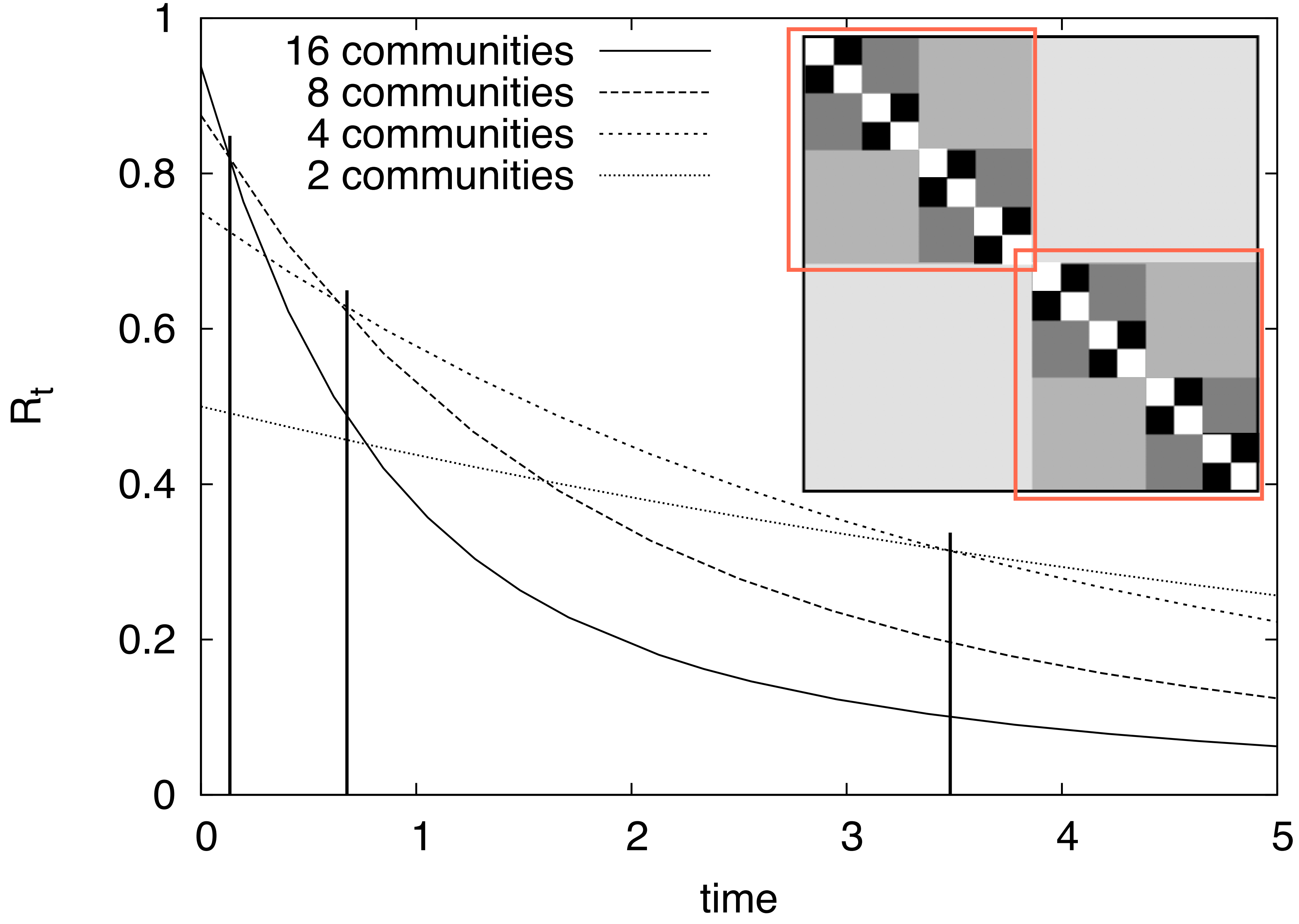
Stability: time as a resolution parameter

Time is a “resolution parameter”: larger and larger communities when time is increased



Stability: time as a resolution parameter

Time is a “resolution parameter”: larger and larger communities when time is increased



nb

The stability *R(t)* of the partition of a graph with adjacency matrix *A* is equivalent to the modularity *Q* of a time-dependent graph with adjacency matrix *X(t)*

$$X_{ij}(t) = \left(e^{t(B-I)}\right)_{ij} k_j$$

$$X_{ij}(t) = X_{ji}(t)$$

which is the flux of probability between 2 nodes at equilibrium and whose generalised degree is

$$\sum_j X_{ij}(t) = k_i$$

$$R(t) = \sum_{i,j} X_{ij}(t)/2m - k_i k_j/(2m)^2 \;\; \delta(c_i, c_j) = Q(X(t))$$

Conclusion

- Relation between dynamics and the hierarchical structure of networks
- Dynamical formulation for the quality of a partition
- Changing time allows to zoom in and out

- Different dynamics lead to different quality functions for the partition of a graph
- Modularity and Stability are radically different in the case of directed networks
- Algorithms developed in order to optimise stability/modularity very large networks

Original Louvain method to optimise modularity available on *http://findcommunities.googlepages.com*

Generalized codes to optimise Q_t available on http://*www.lambiotte.be*

Thanks to J.-L. Guillaume (for providing his c++ code)

R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*

V.D. Blondel, J.-L. Guillaume, R. Lambiotte and E. Lefebvre, J. Stat. Mech., P10008 (2008).